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PLANE AND SPHERICAL
TRIGONOMETRY

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PLANE AND SPHERICAL TRIGONOMETRY

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PREFACE.

This text-book in Trigonometry aims to develop in the student the ability to think out and apply the relations between the trigonometric functions. The definition of the functions by means of projection is not only based on a conception familiar from Geometry, but is also the key to many of the applications of the functions.

Tables of functions are not included in the book, as it is thought better that the student use separate tables.

The student should learn from the start that Trigonometry demands reasoning, not memorizing.

Teachers are beginning to recognize that it is more profitable for students to check computations, either by means of a check formula or by comparing the results of independent work than to rest content with a result that agrees with a printed answer. For this reason answers are as a rule omitted in this text.

TABLE OF CONTENTS.

Part I. Functions of Angles.

Art. I. Angles.
Art. II. Trigonometric functions.
Art. III. Relations between trigonometric functions.
Art. IV. Computation of functions.
Art. V. Applications of functions.
Art. VI. Construction of angles.
Art. VII. Functions of multiples of 90° $\pm A$.
Art. VIII. Functions of $-A$ in functions of A.
Art. IX. Variation of functions.
Art. X. Functions of sum and difference.
Art. XI. Functions of multiple angles.
Art. XII. Functions of half angles.
Art. XIII. The sum and difference of functions expressed as a product.
Art. XIV. Trigonometric equations.
Art. XV. Inverse functions.
Art. XVI. Limits of $\frac{\sin A}{A}$ and $\frac{\tan A}{A}$.

Part II. Solution of Plane Triangles.

Art. XVII. Right triangles.
Art. XVIII. Applications of right triangles.

ART. XIX. Formulas for oblique triangles.
ART. XX. Solution of oblique triangles.
ART. XXI. Applications.

PART III. SPHERICAL TRIGONOMETRY.

ART. XXII. Formulas of spherical trigonometry.
ART XXIII. Solution of right spherical triangles.
ART. XXIV. Solution of oblique spherical triangles.
ART. XXV. Applications.

PLANE AND SPHERICAL TRIGONOMETRY

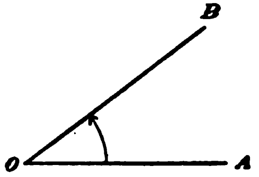
PART I

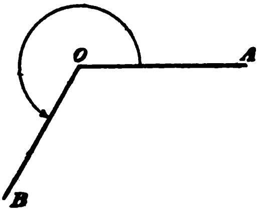
FUNCTIONS OF ANGLES

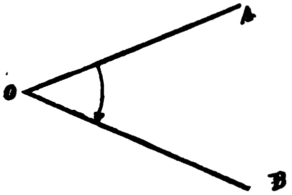
Art. I. Angles.

1. **Definitions.** — A straight line is divided by a point of the line into two parts called half-lines.

A plane angle is the figure formed by two half-lines proceeding from a common point.







The common point is called the vertex of the angle.

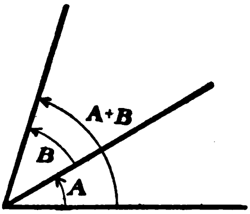
The half-lines are called the sides of the angle.

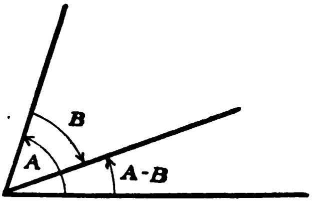
One side OA is called the initial side, the other side OB is called the terminal side of the angle.

The size of the angle is the amount of rotation in the plane of the angle about the vertex which brings the initial side of the angle into coincidence with the terminal side. The amount and direction of this rotation are indicated by a circular arrow, with center at the vertex and extending from the initial side to the terminal side.

If the rotation is anti-clockwise the angle is positive; if the rotation is clockwise the angle is negative.

2. Addition and Subtraction of Angles. — Two positive angles A and B are added by making the terminal side of A the initial side of B and laying off B in the positive direction.

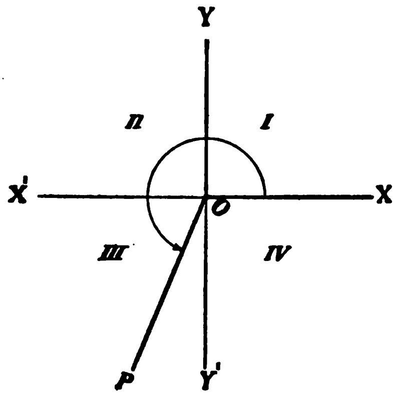




Two positive angles A and B are subtracted by making the terminal side of A the initial side of B and laying off B in the negative direction.

3. The Four Quadrants. — Two straight lines XX' and YY' perpendicular to each other at O divide the plane into four parts called quadrants. XOY is called the first quadrant, YOX' the second quadrant, $X'OY'$ the third quadrant, $Y'OX$ the fourth quadrant.

Take O as the vertex and OX as the initial side of an angle. Let the terminal side start from coincidence with OX and revolve about O in the positive direction. When the terminal side coincides with the initial side the angle is zero; when it coincides with OY the angle XOY is a right angle; when it coincides with OX' the angle XOX' is two right angles; when it coincides with OY' the angle XOY' is three right angles; when it again comes into coincidence with OX the angle is four right angles.



One complete revolution of the terminal side brings it into coincidence with its first position. Hence the terminal sides of all angles differing by multiples of four right angles coincide.

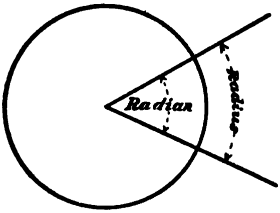
4. **Measurement of Angles.**—Two units are in common use in the measurement of angles—the degree, which is one-ninetieth of a right angle; and the radian, which is the angle at the center of a circle subtended by an arc equal to the radius of the circle.

An angle of 90°, or a right angle, may conveniently be drawn by constructing a triangle whose sides are proportional to 3, 4 and 5. In this triangle the square of the longest side equals the sum of the squares of the other two sides. Hence by Geometry the angle opposite the longest side is a right angle.

An angle of 45° may be constructed by drawing a right angle, laying off equal lines from the vertex on the two sides, and joining the extremities of these lines by a straight line. In the triangle formed the acute angles are 45° angles.

Each angle of an equilateral triangle is an angle of 60°.

An angle of any number of degrees is conveniently laid off by means of a protractor. The student should continually use a divided straight edge and a protractor to draw accurate figures for the graphic solution of problems. The graphic solution furnishes an excellent check on computation.



To find the number of degrees in a radian apply the proposition of geometry—

> The angles at the center of a circle are proportional to their intercepted arcs.

Applying this proposition, the ratio of an angle of four right angles to a radian equals the ratio of the circumference of the circle to the radius. That is

$$\frac{360^\circ}{\text{radian}} = \frac{2\pi R}{R} = 2\pi.$$

Hence.2π radians $= 360^\circ$ and 1 radian $= \dfrac{180^\circ}{\pi} = 57^\circ.3$.

It follows that $1^\circ = \dfrac{2\pi}{360}$ radians $= .017$ radians, $90^\circ = \frac{1}{2}\pi$ radians, $270^\circ = \frac{3}{2}\pi$ radians.

The measure of an angle in radians is called the circular measure of the angle. Therefore the circular measure of a right angle is $\frac{1}{2}\pi$; the circular measure of an angle of 10° is .17. The angle whose circular measure is 2 is an angle of 114°.6.

Again applying the above proposition of geometry, the length of the arc of any circle is to the radius of that circle as the angle subtended by the arc at the center measured in radians is to a radian, that is

$$\frac{\text{arc}}{\text{radius}} = \frac{\theta \text{ radians}}{1 \text{ radian}}.$$

From this it follows that the length of a circular arc is the circular measure of the angle subtended at the center of the circle multiplied by the radius, and also that the circular measure of an angle is the ratio of the arc subtended on the circumference with center at the vertex to the radius.

Notice that the circular measure of an angle is the length of the arc of the angle struck off with radius unity.

Problems. — Draw the following angles

1. 30°. **2.** 15°. **3.** 75°. **4.** 135°. **5.** 150°. **6.** 210°. **7.** 315°.

Find the circular measure of the angles

8. 5° **9.** 30°. **10.** 45°. **11.** 75°. **12.** 120°. **13.** 150°. **14.** 180°. **15.** 200°. **16.** 240°. **17.** 270°. **18.** 300°. **19.** 700°.

Find in degrees, minutes and seconds the angle whose circular measure is

20. $\frac{1}{3}\pi$. **21.** $\frac{3}{2}\pi$. **22.** $\frac{1}{2}$. **23.** $\frac{2}{3}$. **24.** $\frac{4}{5}$. **25.** 3. **26.** 2.6. **27.** 3. **28.** 5.2. **29.** 8.

The radius of a circle is 10 inches. Find the length of the arc which subtends at the center an angle of

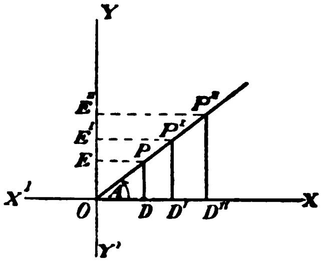
30. 15°. **31.** 40°. **32.** 125°. **33.** $\frac{2}{3}$ radians **34.** $1\frac{1}{2}$ radians.

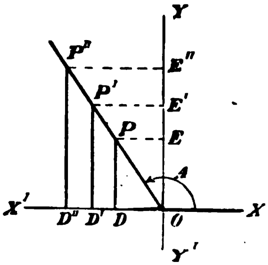
The radius of a circle is 10 inches, find the angle subtended at the center by an arc of

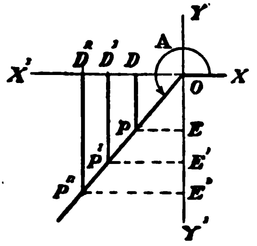
35. 2 inches. **36.** 5 inches. **37.** 8 inches. **38.** 25 inches. **39.** 34 inches.

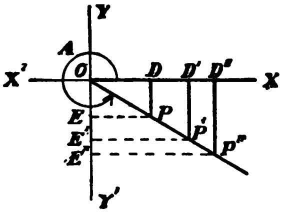
Art. II. Trigonometric Functions.

5. Projections. — Let A be an angle, O its vertex, OX its initial side and OP its terminal side. XX' is the straight line in which the initial side lies. YY' is perpendicular to XX' at the vertex O.









Take any point such as P, P', P'', $\cdots$ in the terminal side of the angle and project OP, OP', OP'', $\cdots$ on XX' and YY'.

The length of the projection on XX' is OD, OD', OD'', $\cdots$. The length of the projection on YY' is OE, OE', OE'', $\cdots$. The projection on XX' is positive if it lies to the right of O, negative if it lies to the left of O. The projection on YY' is positive if it lies above O, negative if it lies below O. The distance OP, OP', OP'', $\cdots$ is always positive.

Notice that the projection of OP on YY' is the distance of P from the initial line XX'. This distance is positive when P lies above XX', negative when P lies below XX'.

If the angle ends in the first quadrant the projections on XX and YY' are both positive; if the angle ends in the second quadrant the projection on XX' is negative, the projection on YY' is positive; if the angle ends in the third quadrant the projections on XX' and YY' are both negative; if the angle ends in the fourth quadrant the projection on XX' is positive, the projection on YY' is negative.

The distance from the vertex of an angle to any point in the terminal line is called the radial distance of that point and is denoted by r.

The projection of the radial distance of a point on the line XX' is called the abscissa of the point and is denoted by x.

The projection of the radial distance of a point on the line YY' is called the ordinate of the point and is denoted by y.

For example, taking the point P' in the terminal side of the angle A ending in the second quadrant, the radial distance is $r = OP'$, the abscissa is $x = -OD'$, the ordinate is $y = OE'$.

6. The Sine. — Whatever point P, P', P'', $\cdots$ be taken in the terminal side of the angle A, the ratio of the projection of the distance of the point from O on YY' to the distance itself is the same. This ratio, which depends only on the size of the angle A is called the sine of the angle A and is written sin A. Hence in the first quadrant

$$\sin A = \frac{OE}{OP} = \frac{OE'}{OP'} = \frac{OE''}{OP''} = \cdots;$$

in the second quadrant

$$\sin A = \frac{OE}{OP} = \frac{OE'}{OP'} = \frac{OE''}{OP''} = \cdots;$$

in the third quadrant

$$\sin A = \frac{-OE}{OP} = \frac{-OE'}{OP'} = \frac{-OE''}{OP''} = \cdots;$$

in the fourth quadrant

$$\sin A = \frac{-OE}{OP} = \frac{-OE'}{OP'} = \frac{-OE''}{OP''} = \cdots.$$

The sine is positive for angles ending in the first and second quadrants, negative for angles ending in the third and fourth quadrants. It will be observed that the sine of a given angle has a fixed numerical value.

The definition of the sine of an angle may be stated as follows —

> The sine of an angle is the ratio of the ordinate of any point in the terminal side of the angle to the radial distance of that point, that is, $\sin A = \frac{y}{r}$.

7. The Cosine. — The ratio of the projection of the distance of P, P', $P'' \cdots$ from O on XX' to the distance itself is called the cosine of A and is written $\cos A$. Hence in the first quadrant

$$\cos A = \frac{OD}{OP} = \frac{OD'}{OP'} = \frac{OD''}{OP''} = \cdots;$$

in the second quadrant

$$\cos A = \frac{-OD}{OP} = \frac{-OD'}{OP'} = \frac{-OD''}{OP''} = \cdots;$$

in the third quadrant

$$\cos A = \frac{-OD}{OP} = \frac{-OD'}{OP'} = \frac{-OD''}{OP''} = \cdots;$$

in the fourth quadrant

$$\cos A = \frac{OD}{OP} = \frac{OD'}{OP'} = \frac{OD''}{OP''} = \cdots.$$

The cosine is positive in the first and fourth quadrants, negative in the second and third quadrants.

The definition of the cosine of an angle may be stated as follows —

> The cosine of an angle is the ratio of the abscissa of any point in the terminal side of the angle to the radial distance of that point, that is $\cos A = \frac{x}{r}$.

8. The Tangent. — The ratio of the projection of the distance on YY' to the projection on XX' is called the tangent of A and is written $\tan A$. Hence in the first quadrant

$$\tan A = \frac{OE}{OD} = \frac{OE'}{OD'} = \frac{OE''}{OD''} = \cdots;$$

in the second quadrant

$$\tan A = \frac{OE}{-OD} = \frac{OE'}{-OD'} = \frac{OE''}{-OD''} = \cdots;$$

in the third quadrant

$$\tan A = \frac{-OE}{-OD} = \frac{-OE'}{-OD'} = \frac{-OE''}{-OD''} = \cdots;$$

in the fourth quadrant

$$\tan A = \frac{-OE}{OD} = \frac{-OE'}{OD'} = \frac{-OE''}{OD''} = \cdots.$$

The tangent is positive in the first and third quadrants, negative in the second and fourth quadrants.

The definition of the tangent of an angle may be stated as follows —

> The tangent of an angle is the ratio of the ordinate of any point in the terminal side of the angle to the abscissa of that point, that is $\tan A = \frac{y}{x}$.

9. The Cotangent. — The ratio of the projection of the distance on XX' to the projection on YY' is called the cotangent of A and is written $\cot A$. Hence in the first quadrant

$$\cot A = \frac{OD}{OE} = \frac{OD'}{OE'} = \frac{OD''}{OE''} = \cdots;$$

in the second quadrant

$$\cot A = \frac{-OD}{OE} = \frac{-OD'}{OE'} = \frac{-OD''}{OE''} = \cdots;$$

in the third quadrant

$$\cot A = \frac{-OD}{-OE} = \frac{-OD'}{-OE'} = \frac{-OD''}{-OE''} = \cdots;$$

in the fourth quadrant

$$\cot A = \frac{OD}{-OE} = \frac{OD'}{-OE'} = \frac{OD''}{-OE''} = \cdots;$$

The cotangent is positive in the first and third quadrants, negative in the second and fourth quadrants.

The definition of the cotangent of an angle may be stated as follows —

> The cotangent of an angle is the ratio of the abscissa of any point in the terminal side of the angle to the ordinate of that point, that is $\cot A = \frac{x}{y}$.

10. The Secant. — The ratio of the distance to its projection on XX' is called the secant of A and is written $\sec A$. Hence in the first quadrant

$$\sec A = \frac{OP}{OD} = \frac{OP'}{OD'} = \frac{OP''}{OD''} = \cdots;$$

in the second quadrant

$$\sec A = \frac{OP}{-OD} = \frac{OP'}{-OD'} = \frac{OP''}{-OD''} = \cdots;$$

in the third quadrant

$$\sec A = \frac{OP}{-OD} = \frac{OP'}{-OD'} = \frac{OP''}{-OD''} = \cdots;$$

in the fourth quadrant

$$\sec A = \frac{OP}{OD} = \frac{OP'}{OD'} = \frac{OP''}{OD''} = \cdots.$$

The secant is positive in the first and fourth quadrants, negative in the second and third quadrants.

The definition of the secant of an angle may be stated as follows —

> The secant of an angle is ratio of the radial distance of any point in the terminal side of the angle to the abscissa of that point, that is $\sec A = \frac{r}{x}$.

11. The Cosecant. — The ratio of the distance to its projection on YY' is called the cosecant of A and is written $\operatorname{cosec} A$. Hence in the first quadrant

$$\operatorname{cosec} A = \frac{OP}{OE} = \frac{OP'}{OE'} = \frac{OP''}{OE''} = \cdots;$$

in the second quadrant

$$\operatorname{cosec} A = \frac{OP}{OE} = \frac{OP'}{OE'} = \frac{OP''}{OE''} = \cdots;$$

in the third quadrant

$$\operatorname{cosec} A = \frac{OP}{-OE} = \frac{OP'}{-OE'} = \frac{OP''}{-OE''} = \cdots;$$

in the fourth quadrant

$$\operatorname{cosec} A = \frac{OP}{-OE} = \frac{OP'}{-OE'} = \frac{OP''}{-OE''} = \cdots.$$

The cosecant is positive in the first and second quadrants, negative in the third and fourth quadrants.

The definition of the cosecant of an angle may be stated as follows —

> The cosecant of an angle is the ratio of the radial distance of any point in the terminal side of the angle to the ordinate of that point, that is $\operatorname{cosec} A = \frac{r}{y}$.

12. **The Trigonometric Functions.** — The six ratios, sine, cosine, tangent, cotangent, secant, and cosecant depend only on the size of the angle and are called the trigonometric functions of the angle.

Two auxiliary functions are defined as follows —

$$\text{versed-sine } A = 1 - \cos A, \quad \text{coversed-sine } A = 1 - \sin A.$$

If an angle is given, the numerical values of the trigonometric functions of the angle can be computed. If the trigonometric functions of an angle are given, the angle can be constructed.

> Trigonometry discusses the values of the trigonometric functions and the relations between them, and shows how to solve problems by means of these functions.

Art. III. Relations between Trigonometric Functions.

13. Consider the angle A ending in the second quadrant. Since OPD is a right triangle,

(*a*) $$(PD)^2 + (OD)^2 = (OP)^2,$$

which may be written

(*b*) $$(PD)^2 + (-OD)^2 = (OP)^2.$$

Dividing (*b*) by $(OP)^2$,

$$\left(\frac{PD}{OP}\right)^2 + \left(\frac{-OD}{OP}\right)^2 = \left(\frac{OP}{OP}\right)^2,$$

that is,

(1) $$\mathbf{\sin^2 A + \cos^2 A = 1.}$$

Dividing (*b*) by $(-OD)^2$,

$$\left(\frac{-OD}{-OD}\right)^2 + \left(\frac{PD}{-OD}\right)^2 = \left(\frac{OP}{-OD}\right)^2,$$

that is,

(2) $$\mathbf{1 + \tan^2 A = \sec^2 A.}$$

Dividing (*b*) by $(PD)^2$

$$\left(\frac{PD}{PD}\right)^2 + \left(\frac{-OD}{PD}\right)^2 = \left(\frac{OP}{PD}\right)^2,$$

that is,

(3) $$\mathbf{1 + \cot^2 A = \operatorname{cosec}^2 A.}$$

By definition $\tan A = \dfrac{PD}{-OD}$. Dividing numerator and denominator by OP, $\tan A = \dfrac{\frac{PD}{OP}}{\frac{-OD}{OP}}$. Hence

(4) $$\mathbf{\tan A = \frac{\sin A}{\cos A}.}$$

In like manner it is proved that

(5) $$\mathbf{\cot A = \frac{\cos A}{\sin A}.}$$

The reciprocal relations

$$(6) \qquad \tan \mathbf{A} = \frac{1}{\cot \mathbf{A}},$$

$$(7) \qquad \sec \mathbf{A} = \frac{1}{\cos \mathbf{A}},$$

$$(8) \qquad \operatorname{cosec} \mathbf{A} = \frac{1}{\sin \mathbf{A}}$$

follow directly from the definitions.

By means of these eight relations, which must be carefully memorized, all functions of an angle may be expressed in terms of any one function. For example,

$$\cos A = \sqrt{1-\sin^2 A}, \quad \tan A = \frac{\sin A}{\sqrt{1-\sin^2 A}}, \quad \cot A = \frac{\sqrt{1-\sin^2 A}}{\sin A},$$

$$\sec A = \frac{1}{\sqrt{1-\sin^2 A}}, \quad \operatorname{cosec} A = \frac{1}{\sin A}.$$

If $\sin A = -\frac{4}{5}$ we find $\cos A = \pm\frac{3}{5}$, $\tan A = \mp\frac{4}{3}$, $\cot A = \mp\frac{3}{4}$, $\sec A = \pm\frac{5}{3}$, $\operatorname{cosec} A = -\frac{5}{4}$.

Problems. — Prove the eight relations for the functions of an angle ending in the

1. First quadrant. 2. The third quadrant. 3. The fourth quadrant.

Express all the functions of an angle in terms of the

1. cosine. 2. tangent. 3. cotangent. 4. secant. 5. cosecant.

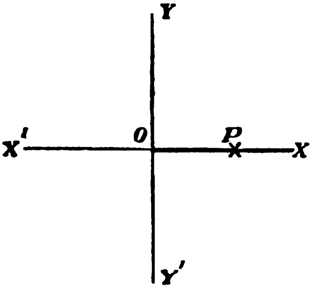
Find all the functions of A when

6. $\sin A = \frac{3}{5}$. 7. $\tan A = -\frac{4}{3}$. 8. $\sec A = 2$.

9. $\operatorname{cosec} A = -\frac{25}{7}$.

Art. IV. Computation of Functions.

14. Functions of Multiples of 90°. — For an angle of 0° the terminal side coincides with the initial side. Hence OP is its own projection on XX' and the projection of OP on YY' is zero. It follows that



$$\sin 0^\circ = \frac{0}{OP} = 0; \qquad \cos 0^\circ = \frac{OP}{OP} = 1;$$

$$\tan 0^\circ = \frac{0}{OP} = 0; \qquad \cot 0^\circ = \frac{OP}{0} = \infty;$$

$$\sec 0^\circ = \frac{OP}{OP} = 1; \qquad \text{cosec}\, 0^\circ = \frac{OP}{0} = \infty;$$

$$\text{vers}\, 0^\circ = 1 - \cos 0^\circ = 0; \quad \text{covers}\, 0^\circ = 1 - \sin 0^\circ = 1.$$

For an angle of 270° the distance of a point P in the terminal side from O is OP, the projection of OP on XX' is zero, the projection of OP on YY' is $-OP$.

Hence

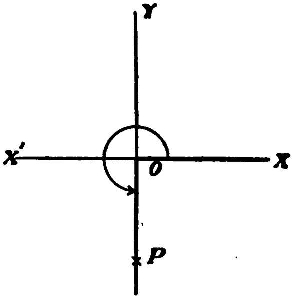
$$\sin 270^\circ = \frac{-OP}{OP} = -1; \qquad \cos 270^\circ = \frac{0}{OP} = 0;$$

$$\tan 270^\circ = \frac{-OP}{0} = -\infty; \qquad \cot 270^\circ = \frac{0}{-OP} = -0;$$

$$\sec 270^\circ = \frac{OP}{0} = \infty; \qquad \operatorname{cosec} 270^\circ = \frac{OP}{-OP} = -1;$$

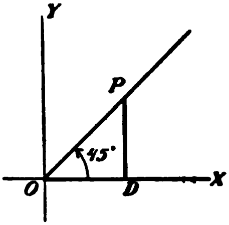
$$\operatorname{vers} 270^\circ = 1 - \cos 270^\circ = 1;$$

$$\operatorname{covers} 270^\circ = 1 - \sin 270^\circ = 2.$$



Problems. — Find the functions of 1. 90°, 2. 180°, 3. 360°, 4. 540°, 5. 450°, 6. 630°.

15. **Functions of Multiples of 45°.** — Construct an angle POX equal to 45°. Lay off OP equal to any distance a, then $OD = PD = \frac{1}{2}\sqrt{2}a$.



Hence

$$\sin 45^\circ = \frac{PD}{OP} = \tfrac{1}{2}\sqrt{2}; \qquad \cos 45^\circ = \frac{OD}{OP} = \tfrac{1}{2}\sqrt{2};$$

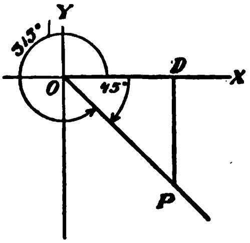
$$\tan 45^\circ = \frac{PD}{OD} = 1; \qquad \cot 45^\circ = \frac{OD}{OP} = 1;$$

$$\sec 45^\circ = \frac{OP}{OD} = \sqrt{2}\,; \qquad \operatorname{cosec} 45^\circ = \frac{OP}{PD} = \sqrt{2}\,;$$

$$\operatorname{vers} 45^\circ = 1 - \tfrac{1}{2}\sqrt{2}\,; \qquad \operatorname{covers} = 1 - \tfrac{1}{2}\sqrt{2}.$$

To find the functions of 315° subtract 45° from 360° and lay off $OP = a$.

Then $OD = PD = \frac{1}{2}\sqrt{2}a$.



Hence

$$\sin 315^\circ = \frac{-PD}{OP} = -\tfrac{1}{2}\sqrt{2}\,; \qquad \cos 315^\circ = \frac{OD}{OP} = \tfrac{1}{2}\sqrt{2}\,;$$

$$\tan 315^\circ = \frac{-PD}{OO} = -1\,; \qquad \cot 315^\circ = \frac{OD}{-PO} = -1\,;$$

$$\sec 315^\circ = \frac{OP}{OO} = \sqrt{2}\,; \qquad \operatorname{cosec} 315^\circ = \frac{OP}{-PD} = -\sqrt{2}\,;$$

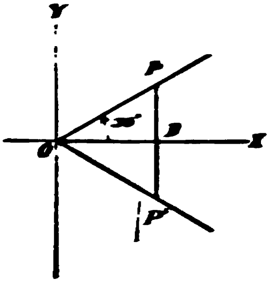
$$\operatorname{vers} 315^\circ = 1 - \tfrac{1}{2}\sqrt{2}\,; \qquad \operatorname{covers} 315^\circ = 1 + \tfrac{1}{2}\sqrt{2}.$$

Problems.—Find the functions of 1. 135°, 2. 225°, 3. 405°, 4. 495°.

16. **Functions of Multiples of 30°.**—Let POX be an angle of 30°. Construct $P'OX$ equal to 30°. Lay off

$$OP = OP' = a, \text{ and draw } PP'.$$

Then $OP = a$, $PD = \frac{1}{2}a$, and $OD = \frac{1}{2}\sqrt{3}a$.



Hence

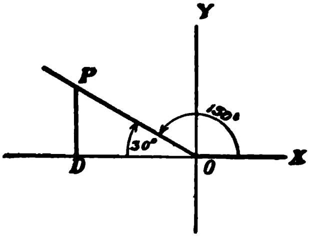
$$\sin 30^\circ = \frac{PD}{OP} = \frac{1}{2}; \qquad \cos 30^\circ = \frac{OD}{OP} = \frac{1}{2}\sqrt{3};$$

$$\tan 30^\circ = \frac{PD}{OD} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}; \qquad \cot 30^\circ = \frac{OD}{OP} = \sqrt{3};$$

$$\sec 30^\circ = \frac{OP}{OD} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}; \qquad \operatorname{cosec} 30^\circ = \frac{OP}{PD} = 2;$$

$$\operatorname{vers} 30^\circ = 1 - \frac{1}{2}\sqrt{3}; \qquad \operatorname{covers} 30^\circ = \frac{1}{2}.$$

To find the functions of 150° subtract an angle of 30° from an angle of 180° and lay off OP equal to a. Then



$$PD = \frac{1}{2}a, \quad OD = \frac{1}{2}\sqrt{3}a.$$

Hence

$$\sin 150^\circ = \frac{PD}{OP} = \frac{1}{2}; \qquad \cos 150^\circ = \frac{OD}{OP} = -\frac{1}{2}\sqrt{3};$$

$$\tan 150^\circ = \frac{PD}{-OD} = -\tfrac{1}{3}\sqrt{3}\,; \quad \cot 150^\circ = \frac{-OD}{OP} = -\sqrt{3}\,;$$

$$\sec 150^\circ = \frac{OP}{-OD} = -\tfrac{2}{3}\sqrt{3}\,; \quad \operatorname{cosec} 150^\circ = \frac{OP}{OD} = 2\,;$$

$$\operatorname{vers} 150^\circ = 1 + \tfrac{1}{2}\sqrt{3}\,; \qquad \operatorname{covers} 150^\circ = \tfrac{1}{2}.$$

Problems. — Find the functions of 1. 60°. 2. 120°. 3. 210°. 4. 240°. 5. 300°. 6. 350°. 7. 420°.

17. Functions of Any Angle. — The functions of multiples of 90°, 45°, and 30° have just been calculated directly from the definitions by the aid of propositions in geometry.

The sine of any angle may be calculated to any required degree of accuracy by means of the series developed in the differential calculus,

$$\sin x = x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\cdots,$$

where x is the circular measure of the angle. For example to calculate sin 10°, $x = .17$.

All the other functions may be obtained from the sine as is shown in Art. III.

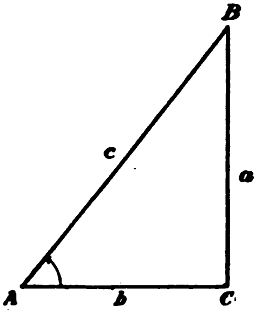
So important are the trigonometric functions of an angle that their values are arranged in tables in a form convenient for reference. The student should procure such a table and learn to use it.

Problems. — Compute and compare results with the tables, 1. sin 5°. 2. cos 9°. 3. sin 12°. 4. sin 18°. 5. sin 20°. 6. sin 40°.

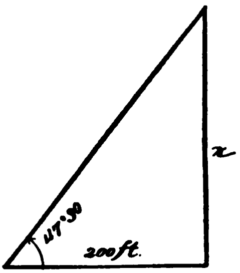
Art. V. Applications of Functions.

18. Many interesting and important problems may be solved by the aid of a table of functions.

Example I. — In a right triangle the hypotenuse is 389, an oblique angle $A = 60° 55'$. Find the sides about the right angle. Since b is the projection of c on AC, by definition $\cos A = \frac{b}{c}$ whence $b = c \cdot \cos A = 389 \times .48608 = 189.1$; by definition $\sin A = \frac{a}{c}$ whence $a = c \cdot \sin A = 389 \times .87391 = 340.$



Example II. — The angle of elevation to the top of a steeple is 47° 30′ measured from a point in the horizontal through the base and 200 feet from the base. What is the height of the steeple?



By definition $\tan 47^\circ\, 30' = \frac{x}{200}$, whence

$$x = 200 \times \tan 47^\circ\, 30' = 218.26 \text{ feet.}$$

Problems. — 1. In a right triangle the hypotenuse is 100, an oblique angle 57° 45′. Find the other two sides.

2. The sides about the right angle are 8 and 17. Find the oblique angles of the triangle.

3. The angle at the vertex of a right cone is 52° 24′ and the slant height is 126 ft. Find diameter of the base and the altitude.

4. The legs of a pair of dividers are so set that the angle between them is 80° 24′. The legs are six inches long. What is the distance between the points?

5. The ridgepole of a roof is 15 feet above the center of the garret floor and the garret is 40 feet wide. What is the inclination of the roof to the horizontal?

6. Find the side and the apothem of a regular decagon inscribed in a circle whose radius is 10 feet.

7. The side of a regular decagon is 12 feet. Find the radii of inscribed and circumscribed circles.

8. Find the perimeter of the equilateral triangle circumscribed about the circle whose radius is 15 feet.

9. A tower 103 feet high throws a shadow 51.5 feet long on the horizontal plane of the base. What is the angle of elevation of the sun?

10. A railroad track has a uniform grade of 3°. Find the vertical rise of 463 feet 3 inches of track.

11. Find the length of the diagonal of a regular pentagon whose side is 20 feet.

12. From the top of a cliff 100 feet above sea-level the angles of depression of two ships are found to be 45° and 30°. The observer and the ships being in a straight line, find the distance between the ships.

13. What distance in space is travelled in an hour in consequence of the earth's rotation by a person in latitude 40°?

14. Upon the top of a shaft 125 feet high stands a statue which, at a horizontal distance of 200 feet from the foot of the shaft, subtends an angle of 3°. How tall is the statue?

15. From a boat the angles of elevation of the highest and lowest points of a flagstaff, 30 feet high, on the edge of a cliff are observed to be 46° 12′ and 44° 13′. Determine the height of the cliff and its distance.

16. A man in a balloon when it is one mile high finds the angle of depression of an object on the level ground to be 35° 20′. After ascending vertically and uniformly for 20 minutes he finds the angle of depression of the same object to be 55° 40′. Find the rate of ascent of the balloon in miles per hour.

17. Show that the projection of one line-segment on another is always the given line-segment times the cosine of the angle between the line-segments.

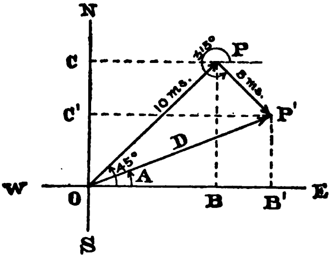
18. A man walks 20 miles N. E. How far has he walked towards the east and how far towards the north?

19. Two sides of a parallelogram, 60 feet and 80 feet, include an angle of 50°. Find the length of the diagonal of the parallelogram through the given angle and the inclination of this diagonal to the side 80 feet long.

20. Show that the sum of the projections of the parts of a broken line on any straight line is the projection on that straight line of the line joining the ends of the broken line.

21. A man walks N. E. 10 miles, then S. E. 5 miles. What is his distance and bearing from the starting point?

Suggestion. — Draw an accurate figure. Suppose the man to start from O, to reach P after walking N. E. 10 miles, and to reach P' after walking 5 miles S. E. from P. Let D represent the length of OP', A the angle $P'OE$. The problem is solved when D and A are found.



OP makes with OE an angle of 45°, PP' makes with OE an angle of 315°.

The projection of D on OE is

$$D \cos A = OB' = OB + BB' = 10 \cos 45° + 5 \cos 315° = 15 \cos 45°.$$

The projection of D on ON is

$$D \sin A = OC' = OC - CC' = 10 \sin 45° + 5 \sin 315° = 5 \sin 45°.$$

From the equations (1) $D \sin A = 5 \sin 45°$ and (2) $D \cos A = 15 \cos 45°$, D and A can be computed.

Adding the squares of (1) and (2)

$$D^2 = 25 \sin^2 45° + 225 \cos^2 45° = 250 \sin^2 45° = 125,$$

and $D = \sqrt{125} = 11.18$. That is, the man is 11.18 miles from the starting point.

Dividing (1) by (2), $\tan A = \frac{1}{3}$ and by referring to the table of trigonometric functions A is found to be 18° 25′. That is, the direction of the man's position from the starting point is found by facing towards the east and turning through an angle 18° 25′ towards the north. This is called the bearing of P' from O and is written E. 18° 25′ N.

22. A man walks N. 8 miles, then E. 4 miles. What is his distance and bearing from the starting point?

23. Three sides of a quadrilateral taken in order are 11 feet, 7 feet, and 8 feet and make angles of 18° 18′, 74° 50′, and 130° 20′ respectively with a fixed line. Find the fourth side of the quadrilateral and the angle this side makes with the fifth line. Ans., 17.79 feet; 66° 30′.

24. The following are the field notes of a survey:

Stations	Bearings	Distances
A	N. 31½° W.	10 chains
B	N. 62¾° E.	9.25
C		
D	S. 45½° W.	10.40

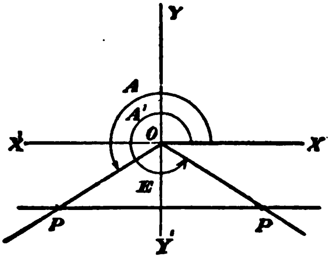
What is the bearing and distance from station C to D?
Ans., S. 38° 52′ E.; 7.03 chains.

25. A man walks 8, 12, 15, and 20 miles along successive straight lines making angles respectively of 30°, 70°, 120° 15′, and 155° with the W. E. line. Find the distance and bearing of his final position from the starting point.
Ans., 39.5 miles; N. 21° 45′ W.

Art. VI. Construction of Angles.

19. To construct the angle whose sine is $-\frac{3}{5}$.

Draw lines XX', YY' perpendicular to each other at O. Take O as the vertex, OX as the initial side of the required angle. If P is a point in the terminal side of the required angle, the projection of OP on YY' must fall below O and this projection must bear to OP the ratio of 3 to 5. Lay off downwards from O a line OE three times any convenient length a, and draw through E a line parallel to XX'. The point P must lie in this line. The point P is at a distance $5a$ from O. Hence the point P must also lie in the circumference struck off from O as a center with a radius of $5a$. Two points P are thus located, determining two angles whose sine is $-\frac{3}{5}$, A ending in the third quadrant and A' ending in the fourth quadrant.



After constructing the angles A and A' all the functions of the angles can be read off from the figure.

For angle A,

$$\cos A = \frac{-PE}{OP} = -\tfrac{4}{5}; \qquad \tan A = \frac{-OE}{-PE} = \tfrac{3}{4};$$

$$\cot A = \frac{-PE}{-OE} = \tfrac{4}{3}; \qquad \sec A = \frac{OP}{-PE} = -\tfrac{5}{4};$$

$$\operatorname{cosec} A = \frac{OP}{-OE} = -\tfrac{5}{3}; \qquad \operatorname{vers} A = 1 - \cos A = 1\tfrac{4}{5};$$

$$\operatorname{covers} A = 1 - \sin A = 1\tfrac{3}{5}.$$

For angle A',

$$\cos A' = \frac{P'E}{OP'} = \tfrac{4}{5}; \qquad \tan A' = \frac{-OE}{P'E} = -\tfrac{3}{4};$$

$$\cot A' = \frac{P'E}{-OE} = -\tfrac{4}{3}; \qquad \sec A' = \frac{OP}{P'E} = \tfrac{5}{4};$$

$$\operatorname{cosec} A' = \frac{OP'}{-OE} = -\tfrac{5}{3}; \qquad \operatorname{vers} A' = 1 - \cos A' = \tfrac{1}{5};$$

$$\operatorname{covers} A' = 1 - \sin A' = 1\tfrac{3}{5}.$$

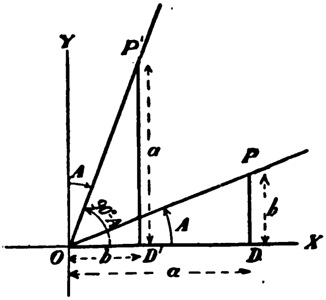
Problems. — In each of the following construct the angle and find the values of the other functions.

1. $\cos A = -\frac{3}{5}$.
2. $\operatorname{cosec} A = \frac{13}{5}$.
3. $\tan A = -\frac{24}{7}$.
4. $\cot A = \frac{4}{3}$.
5. $\tan A = \frac{12}{5}$.
6. $\cos A = -\frac{21}{39}$.
7. $\tan A = 2$.
8. $\operatorname{cosec} A = 3$.
9. $\sin A = \frac{1}{3}$.
10. $\tan A = 3$.

Art. VII. Functions of Multiples of 90° ± A.

20. Functions of $90° - A$ in Functions of A.

Construct the angles A and $90° - A$. Lay off



$$OP = OP' = c.$$

Draw PD and $P'D'$ perpendicular to XX'.
The triangles OPD, $OP'D'$ are equal.

Hence $PD = OD' = b$, $OD = P'D' = a$. By definition

$$\sin(90° - A) = \frac{a}{c} = \cos A\,; \quad \cos(90° - A) = \frac{b}{c} = \sin A\,;$$

$$\tan(90° - A) = \frac{a}{b} = \cot A\,; \quad \cot(90° - A) = \frac{b}{a} = \tan A\,;$$

$$\sec(9° - A) = \frac{c}{b} = \operatorname{cosec} A\,;$$

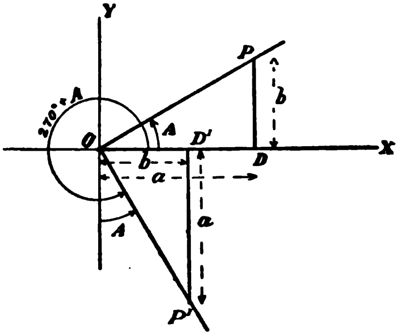
$$\operatorname{cosec}(90° - A) = \frac{c}{a} = \sec A.$$

Observe that the cosine of A is the sine of the complement of A. In general, the co-functions are the functions of the complement. This explains the origin of the terms cosine, cotangent, cosecant.

21. Functions of $270° + A$ in Functions of A.

Construct the angles A and $270° + A$. Lay off

$$OP = OP' = c.$$



Draw PD and $P'D'$ perpendicular to XX'.
The triangles OPD, $OP'D'$ are equal.

Hence $PD = OD' = b$, $OD = P'D' = a$. By definition

$$\sin(270^\circ + A) = \frac{-a}{c} = -\cos A;$$

$$\cos(270^\circ + A) = \frac{b}{c} = \sin A;$$

$$\tan(270^\circ + A) = \frac{-a}{b} = -\cot A;$$

$$\cot(270^\circ + A) = \frac{b}{-a} = -\tan A;$$

$$\sec(270^\circ + A) = \frac{c}{b} = \operatorname{cosec} A;$$

$$\operatorname{cosec}(270^\circ + A) = \frac{c}{-a} = -\sec A.$$

The functions of any multiple of $90^\circ \pm A$ are found precisely as the functions of $90^\circ - A$ and $270^\circ + A$ have been found. Draw the angle A, the multiple of 90°, and the angle whose functions are to be found. From points in the terminal lines of the angle A and the multiple of $90^\circ \pm A$ draw perpendiculars to XX'. Two equal triangles are found from which the functions of the two angles are read off.

Problems.—Derive in functions of A the functions of

1. $90^\circ + A$. 2. $180^\circ - A$. 3. $180^\circ + A$. 4. $270^\circ - A$.
5. $360^\circ - A$. 6. $450^\circ + A$. 7. $630^\circ - A$. 8. $720^\circ + A$.

The function of any multiple of $90^\circ \pm A$ is numerically the same function of the angle A if we have an even multiple of 90°, and the corresponding co-function of the angle if we have an odd multiple of 90°. In either case the algebraic sign of the function must be determined by observing the quadrant in which the angle ends.

For example

$$\tan 325^\circ = \tan(270^\circ + 55^\circ) = -\cot 55^\circ.$$

The number of whole quadrants is three, and tangent is replaced by cotangent. The angle 325° ends in the fourth quadrant where the tangent is negative.

Notice that the tables give the values of the functions from 0° to 90°. The above rule is used to find functions of angles greater than 90° in terms of functions of angles less than 90°.

Problems. — Find in functions of acute angles

1. $\cot 155^\circ$. **2.** $\sec 457^\circ$. **3.** $\tan 205^\circ$. **4.** $\cos 320^\circ$.

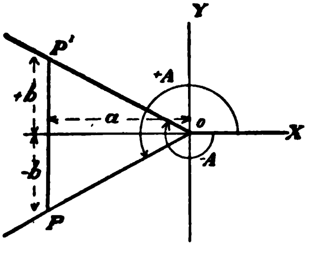
5. $\sin 475^\circ$. **6.** $\tan 700^\circ$. **7.** $\sin 500^\circ$. **8.** $\operatorname{cosec} 400^\circ$.

Art. VIII. Functions of $-A$ in Functions of A.

22. Let A be a positive angle ending in the third quadrant and construct an equal negative angle. Lay off

$$OP = OP' = c.$$

The projection of c on X' is $-a$, the projection of OP on YY' is $-b$, of OP' on YY' is $+b$. By definition



$$\sin(-A) = \frac{+b}{c} = -\sin A; \quad \cos(-A) = \frac{-a}{c} = \cos A;$$

$$\tan(-A) = \frac{+b}{-a} = -\tan A;\quad \cot(-A) = \frac{-a}{+b} = -\cot A;$$

$$\sec(-A) = \frac{c}{-a} = \sec A;\quad \operatorname{cosec}(-A) = \frac{c}{+b} = +\operatorname{cosec} A.$$

Problems.—Show effect of change of sine of angle on functions of angle when angle ends in

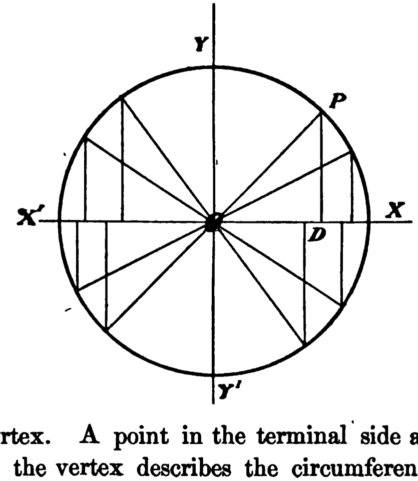
1. second quadrant. 2. first quadrant. 3. fourth quadrant.

Notice that all the results of this article are contained in the rule—

Changing the sign of an angle changes the sign of the sine, tangent, cotangent, and cosecant but has no effect on cosine and secant.

Art. IX. Variation of the Functions.

23. The Sine.—Let the terminal side of an angle, starting from coincidence with the initial side, make one revolution about the vertex. A point in the terminal side at a fixed distance c from the vertex describes the circumference of a circle



with the center at the vertex and radius c. As the angle increases from 0° to 90° the projection of OP on YY' increases from 0 to c and the sine increases from 0 to $+1$. As the angle increases from 90° to 180° the projection on YY' decreases from c to 0 and the sine decreases from $+1$ to 0. As the angle increases from 180° to 270° the projection on YY' decreases from -0 to $-c$ and the sine decreases from -0 to -1. As the angle increases from 270° to 360° the projection on YY' increases from $-c$ to -0 and the sine increases from -1 to -0.

The sine can have any value from $+1$ to -1 through 0.

The sine of 180° is $= +0$ or -0 according as 180° is considered as belonging to the second or the third quadrant.

24. The Tangent. — As the angle increases from 0° to 90° the projection on YY' increases from 0 to c, the projection on XX' decreases from c to 0, hence the tangent increases from 0 to $+\infty$.

As the angle increases from 90° to 180° the projection on YY' decreases from c to 0, the projection on XX' decreases from 0 to $-c$, hence the tangent increases from $-\infty$ to -0.

As the angle increases 180° to 270° the projection on YY' decreases from -0 to $-c$, the projection on XX' increases from $-c$ to -0, hence the tangent increases from 0 to ∞.

As the angle increases from 270° to 360° the projection on YY' increases from $-c$ to -0, the projection on XX' from -0 to $-c$, hence the tangent increases from $-\infty$ to -0.

The tangent can have any value from $+\infty$ to $-\infty$ through o.

Problems. — Discuss the variation of the

1. cosine. 2. co-tangent. 3. secant.
4. cosecant. 5. versed-sine. 6. coversed-sine.

25. Notice that from 0° to 90° an increase in the angle causes an increase in the sine, tangent and secant, a decrease in

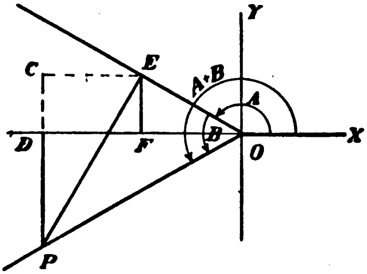
cosine, cotangent and cosecant. For this reason in using the tables the corrections for seconds are added in the case of the direct functions, subtracted in the case of the cofunctions.

Notice also by examining the tables that the sine, tangent and cotangent at the beginning of the quadrant, and the cosine, tangent and cotangent at the end of the quadrant change much more rapidly than in other parts of the quadrant. For this reason the method of making corrections for seconds which is satisfactory in other parts of the quadrant is not satisfactory at the beginning and end of the quadrant. This makes a special table necessary for functions of angles near 0° and near 90°.

Art. X. Functions of the Sum and Difference.

26. To prove that

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B, \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B. \end{aligned} \tag{9}$$



Let A be greater than 90° and less than 180°, B less than 90°, and $A+B$ greater than 180°.

Construct the angle A on OX as the initial side. Then construct the angle B using the terminal side of A as initial side of B. OX is the initial side of $A+B$, OP the terminal side.

From any point P in the terminal side of $A + B$ draw a perpendicular PD to the initial line XX'. From P, which is also a point in the terminal side of B, draw a perpendicular PE to the initial side of B. From E, which is a point in the terminal side of A, draw a perpendicular EF to the initial line of A. Draw EC perpendicular to PD produced. The triangles OEF and PCE are similar.

By definition

$$\sin (A + B) = -\frac{PD}{OP}, \quad \cos (A + B) = -\frac{OD}{OP}.$$

$$\sin A = \frac{EF}{OE} = \frac{CE}{PE}, \quad \cos A = \frac{-OF}{OE} = \frac{-PC}{PE}.$$

$$\sin B = \frac{PE}{OP}, \quad \cos B = \frac{OE}{OP}.$$

(*a*) From the figure $PD = PC - EF$, hence

$$\sin (A + B) = \frac{-PD}{OP} = \frac{EF}{OP} - \frac{PC}{OP}.$$

Multiplying numerator and denominator of

$$\frac{EF}{OP} \text{ by } OE, \quad \frac{EF}{OP} = \frac{EF}{OE} \cdot \frac{OE}{OP};$$

multiplying numerator and denominator of

$$\frac{PC}{OP} \text{ by } PE, \quad \frac{PC}{OP} = \frac{PC}{PE} \cdot \frac{PE}{OP}.$$

Hence $\frac{-PD}{OP} = \frac{EF}{OE} \cdot \frac{DE}{OP} - \frac{PC}{PE} \cdot \frac{PE}{OP}$, that is

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

(*b*) From the figure $OD = OF + EC$, hence

$$\cos (A + B) = \frac{-OD}{OP} = \frac{-OF}{OP} - \frac{EC}{OP}.$$

Multiplying numerator and denominator of

$$\frac{OF}{OP} \text{ by } OE, \quad \frac{OF}{OP} = \frac{OF}{OE} \cdot \frac{OE}{OP};$$

multiplying numerator and denominator of

$$\frac{EC}{OP} \text{ by } PE, \quad \frac{EC}{OP} = \frac{EC}{PE} \cdot \frac{PE}{OP}.$$

Hence $\frac{-OD}{OP} = -\frac{OF}{OE} \cdot \frac{OE}{OP} - \frac{EC}{PE} \cdot \frac{PE}{OP}$ that is,

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

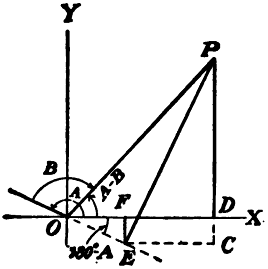
Problems. — Demonstrate the formulas for $\sin(A + B)$ and $\cos(A + B)$ when

1. $A < 90°$, $B < 90°$, $A + B < 90°$.
2. $A < 90°$, $B < 90°$, $A + B > 90°$.
3. $90° < A < 180°$, $B < 90°$, $A + B < 180°$.
4. $90° < A < 180°$, $90° < B < 180°$, $A + B < 270°$.
5. $90° < A < 180°$, $90° < B < 180°$, $270° < A + B < 360°$.

27. To prove that

$$\begin{aligned} \sin(\mathbf{A} - \mathbf{B}) &= \sin \mathbf{A} \cos \mathbf{B} - \cos \mathbf{A} \sin \mathbf{B}, \\ \cos(\mathbf{A} - \mathbf{B}) &= \cos \mathbf{A} \cos \mathbf{B} + \sin \mathbf{A} \sin \mathbf{B}. \end{aligned} \tag{10}$$

Let A be greater than 90° and less than 180°, B greater than 90° but less than A.



From any point P in the terminal side of $A - B$ draw a perpendicular PD to the initial side OX. From P, which is also a point in the terminal side of B, draw a perpendicular PE to the initial line of B. From E, which is a point in the terminal side of $180° - A$, draw a perpendicular EF to OX. Draw EG perpendicular to PD produced. The triangles OEF and PCE are similar.

By definition

$$\sin(A - B) = \frac{PD}{OP}, \quad \cos(A - B) = \frac{OD}{OP},$$

$$\sin A = \sin(180° - A) = \frac{EF}{OE} = \frac{EC}{PE},$$

$$\cos A = -\cos(180° - A) = -\frac{OF}{OE} = -\frac{PC}{PE},$$

$$\sin B = \frac{PE}{PO}, \quad \cos B = \frac{-OE}{PO},$$

(*a*) From the figure $PD = PC - EF$, hence

$$\sin(A - B) = \frac{PD}{OP} = \frac{PC}{OP} - \frac{EF}{OP}.$$

Multiplying numerator and denominator of

$$\frac{PC}{OP} \text{ by } PE, \quad \frac{PC}{OP} = \frac{PC}{PE} \cdot \frac{PE}{OP};$$

multiplying numerator and denominator of

$$\frac{EF}{OP} \text{ by } OE, \quad \frac{EF}{OP} = \frac{EF}{OE} \cdot \frac{OE}{OP}.$$

Hence $\frac{PD}{OP} = -\frac{EF}{OE} \cdot \frac{OE}{OP} + \frac{PC}{PE} \cdot \frac{PE}{OP}$, that is

$$\sin(A - B) = \sin A \cos B - \cos A \sin B,$$

(*b*) From the figure $OD = OF + EC$, hence

$$\cos(A - B) = \frac{OD}{OP} = \frac{OF}{OP} + \frac{EC}{OP}.$$

Multiplying numerator and denominator of

$$\frac{OF}{OP} \text{ by } OE, \quad \frac{OF}{OP} = \frac{OF}{OE} \cdot \frac{OE}{OP};$$

multiplying numerator and denominator of

$$\frac{EC}{OP} \text{ by } PE, \quad \frac{EC}{OP} = \frac{EC}{PE} \cdot \frac{PE}{OP}.$$

Hence $\frac{OD}{OP} = \frac{OF}{OE} \cdot \frac{OE}{OP} + \frac{EC}{PE} \cdot \frac{PE}{OP}$, that is

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Problems. — Demonstrate the formulas for $\sin(A - B)$ and $\cos(A - B)$ when

1. $A < 90°$, $A > B$.
2. $90° < A < 180°$, $A - B < 90°$.
3. $180° < A < 270°$, $B > 90°$, $A - B > 90°$.
4. $180° < A < 270°$, $B > 90°$, $A - B < 90°$.
5. $A > 270°$, $B > 90°$, $A - B > 90°$.

28. To prove that

$$\tan(\mathbf{A} + \mathbf{B}) = \frac{\tan \mathbf{A} + \tan \mathbf{B}}{1 - \tan \mathbf{A} \tan \mathbf{B}};$$

$$\tan(\mathbf{A} - \mathbf{B}) = \frac{\tan \mathbf{A} - \tan \mathbf{B}}{1 + \tan \mathbf{A} \tan \mathbf{B}};$$

(11)

$$\cot(\mathbf{A} + \mathbf{B}) = \frac{\cot \mathbf{A} \cot \mathbf{B} - 1}{\cot \mathbf{B} + \cot \mathbf{A}};$$

$$\cot(\mathbf{A} - \mathbf{B}) = \frac{\cot \mathbf{A} \cot \mathbf{B} + 1}{\cot \mathbf{B} - \cot \mathbf{A}}.$$

Since tangent $= \dfrac{\text{sine}}{\text{cosine}}$,

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing numerator and denominator by $\cos A \cos B$,

$$\tan (A + B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the other three formulas in the same manner.

Problems. — Find the functions of 1. 75°. 2. 15°.

3. Prove $\dfrac{\sin (x + y)}{\sin (x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.

4. Prove $\dfrac{\cos (x + y)}{\cos (x - y)} = \dfrac{1 - \tan x \tan y}{1 + \tan x \tan y}$.

5. Prove $\dfrac{\sin (2x)}{\sin x} - \dfrac{\cos (2x)}{\cos x} = \sec x$.

6. Prove $\sin (45° + x) + \sin (45° - x) = \sqrt{2} \cos x$.

7. Prove $\tan A + \tan B = \dfrac{\sin (A + B)}{\cos A \cos B}$.

8. Prove $\cot A + \tan B = \dfrac{\cos (A - B)}{\sin A \cos B}$.

9. Prove $\cos (A + 45°) + \sin (A - 45°) = 0$.

10. Prove $\dfrac{\sin (3A)}{\sin A} - \dfrac{\cos (3A)}{\cos A} = 2$.

11. From the equation $\sin (A + x) = m \sin x$ find x in terms of A and m.

Suggestion. — Expanding $\sin(A+x)$ and factoring,

$$\begin{aligned}\sin(A+x) &= \sin A\cos x + \cos A\sin x\\ &= \sin A\sin x(\cot x + \cot A).\end{aligned}$$

Hence

$$\sin A(\cot x + \cot A) = m \quad \text{and} \quad \cot x = \frac{m}{\sin A} - \cot A.$$

If the numerical values of m and A are known, the numerical value of $\cot x$ becomes known and the angle x can be found in degrees by referring to the tables of trigonometric functions.

12. From $\sin(A-x) = m\sin x$ find x.

13. From the equation $\tan(A+x) = m\tan x$ find x in terms of A and m.

Suggestion.

The given equation may be written

$$\frac{\tan(A+x)}{\tan x} = \frac{m}{1},$$

whence by composition and division

$$\frac{\tan(A+x)+\tan x}{\tan(A+x)-\tan x} = \frac{m+1}{m-1}.$$

Substituting $\frac{\sin}{\cos}$ for tan and simplifying,

$$\frac{\sin(A+x)\cos x + \cos(A+x)\sin x}{\sin(A+x)\cos x - \cos(A+x)\sin x} = \frac{\sin(2A+x)}{\sin A} = \frac{m+1}{m-1}.$$

Finally $\sin(2A+x) = \frac{m+1}{m-1}\sin A.$

If the numerical values of m and A are known, the numerical value of $\sin(2A+x)$ becomes known and the angle $2A+x$ can be taken from the tables. Since A is given the angle x becomes known.

14. Find x from $\tan(A + x)\tan x = m$.

15. From the equation $m\cos x + n\sin x = q$ find x in terms of m, n and q.

Suggestion. — Place $m = k\sin\varphi$, $n = k\cos\varphi$, whence

$$k = \sqrt{m^2 + n^2} \quad \text{and} \quad \tan\varphi = \frac{m}{n}.$$

The given equation becomes

$$k\sin\varphi\cos x + k\cos\varphi\sin x = q$$

or

$$k\sin(\varphi + x) = q \quad \text{and} \quad \sin(\varphi + x) = \frac{q}{k}.$$

If the numerical values of m, n and q are known, the numerical values of k and φ can be found. Then the numerical value of $\sin(\varphi + x)$ can be found and $\varphi + x$ becomes known. When $\varphi + x$ and φ are known x can be found.

Art. XI. Functions of Multiple Angles.

29. In the formulas for $\sin(A+B)$, $\cos(A+B)$, $\tan(A+B)$, and $\cot(A + B)$ place $B = A$. The formulas become

$$\sin(2A) = 2\sin A\cos A;$$

$$(12) \quad \cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A;$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}; \quad \cot(2A) = \frac{\cot^2 A - 1}{2\cot A}.$$

In the formula

$$\sin(A + B) = \sin A\cos B + \cos A\sin B,$$

place $B = 2A$. There results

$$\sin(3A) = \sin A\cos(2A) + \cos A\sin(2A)$$

$$= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A\cos A)$$

$$= \sin A - 2\sin^3 A + 2\sin A\,(1 - \sin^2 A)$$
$$= 3\sin A - 4\sin^3 A.$$

Problems. — Prove 1. $\cos(3A) = 4\cos^3 A - 3\cos A$.

2. $\tan(3A) = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

3. $\cot(3A) = \dfrac{\cot 3A - 3\cot A}{3\cot^2 A - 1}$.

4. $\sin(4A) = 4\sin A\cos A - 8\sin^3 A\cos A$.

5. $\cos(4A) = 1 - 8\cos^2 A + 8\cos^4 A$.

6. $\cos^4 A - \sin^4 A = \cos(2A)$.

7. $\sin(2A) = \dfrac{2\tan A}{1 + \tan^2 A}$.

8. $\cos(2A) = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$.

9. $\dfrac{\cos(2A)}{1 + \sin(2A)} = \dfrac{1 - \tan A}{1 + \tan A}$.

10. $\dfrac{\sin A + \sin(2A)}{1 + \cos A + \cos(2A)} = \tan A$.

11. In a right triangle whose sides are 3, 4 and 5 a straight line is drawn from the vertex of the right angle to the middle point of the hypotenuse. Find the angle which this line makes with the hypotenuse.

Art. XII. Functions of Half Angles.

30. In the formula

$$\cos(2A) = 1 - 2\sin^2 A \quad \text{place} \quad A = \tfrac{1}{2}B.$$

There results $\cos B = 1 - 2\sin^2(\tfrac{1}{2}B)$. Solving for $\sin\tfrac{1}{2}B$,

(13) $$\sin \tfrac{1}{2}\mathbf{B} = \sqrt{\tfrac{1}{2}(1 - \cos \mathbf{B})}.$$

In like manner $\cos \frac{1}{2}B = \sqrt{\frac{1}{2}(1 + \cos B)}$.

It follows that

$$\tan \tfrac{1}{2}B = \frac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B} = \sqrt{\frac{1 - \cos B}{1 + \cos B}} = \sqrt{\frac{1 - \cos^2 B}{(1 + \cos B)^2}}$$
$$= \sqrt{\frac{\sin^2 B}{(1 + \cos B)^2}} = \frac{\sin B}{1 + \cos B}.$$

Problems. — 1. Prove $\cot \frac{1}{2}B = \dfrac{\sin B}{1 - \cos B}$.

2. Find functions of $22\frac{1}{2}°$.

3. Prove $\dfrac{\cos A}{1 - \sin A} = \dfrac{\cot \frac{1}{2}A + 1}{\cot \frac{1}{2}A - 1}$.

4. Prove $\left(\sin \dfrac{A}{2} + \cos \dfrac{A}{2}\right)^2 = 1 + \sin A$.

5. Prove $\dfrac{1 + \sin A + \cos A}{1 + \sin A - \cos A} = \cot \dfrac{A}{2}$.

Art. XIII. The Sum and Difference of Functions Expressed as a Product.

31. The sum of

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

and

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$

is

(*a*) $$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B.$$

Now place $A + B = x$, $A - B = y$, whence $A = \frac{1}{2}(x + y)$ and $B = \frac{1}{2}(x - y)$. Equation (*a*) becomes

(14) $$\sin \mathbf{x} + \sin \mathbf{y} = 2 \sin \tfrac{1}{2}(\mathbf{x} + \mathbf{y}) \cos \tfrac{1}{2}(\mathbf{x} - \mathbf{y}),$$

that is the sum of the sines of two angles is twice the product of the sine of one-half the sum of the angles and the cosine of one-half the difference of the angles.

Problems. — Prove

1. $\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$.

2. $\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$.

3. $\cos x - \cos y = 2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(y - x)$.

4. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan \frac{1}{2}(x + y)}{\tan \frac{1}{2}(x - y)}$.

5. $\dfrac{\sin (5x) + \sin x}{\cos (5x) + \cos x} = \tan (3x)$.

6. $\sin 50° + \sin 10° = \sin 70°$.

7. $\dfrac{\sin (3A) + \sin (2A)}{\cos (2A) - \cos (3A)} = \cot \dfrac{A}{2}$.

8. $\dfrac{\sin A - \sin B}{\cos B - \cos A} = \cot \dfrac{A + B}{2}$.

9. $\sin x \sin y = \tfrac{1}{2}\{\cos (x - y) - \cos (x + y)\}$.

10. From the equation $\sin (A + x) \sin x = m$ find x in terms of A and m.

Suggestion.

$$\sin (A + x) \sin x = \tfrac{1}{2}\{\cos A - \cos (A + 2x)\} = m.$$

Whence $\cos (A + 2x) = \cos A - 2m$, which determines $A + 2x$ and x becomes known.

11. From $\cos (A - x) \cos x = m$ find x in terms of m and A.

12. From $\cos (A + x) \sin x = m$ find x in terms of m and A.

Art. XIV. Trigonometric Equations.

32. Example I. — Solve $\sin(2x) + \cos x = 0$.

Replacing $\sin(2x)$ by $2 \sin x \cos x$, the equation becomes $2 \sin x \cos x + \cos x = 0$ or $\cos x\,(2 \sin x + 1) = 0$. This equation is satisfied when $\cos x = 0$ and also when $2 \sin x + 1 = 0$. When $\cos x = 0$, x must be an odd multiple of 90° or $\frac{1}{2}\pi$. Hence

$$x = (2n + 1)90^\circ \quad \text{or} \quad x = (2n + 1)\frac{\pi}{2},$$

where n is any whole number. When $2 \sin x + 1 = 0$, $\sin x = -\frac{1}{2}$ and x is an odd multiple of 180° plus 30° or an even multiple of 180° minus 30°. Hence

$$x = (2n + 1)180^\circ + 30^\circ \quad \text{or} \quad x = (2n + 1)\pi + \tfrac{1}{6}\pi$$

and

$$x = 2n\,180^\circ - 30^\circ \quad \text{or} \quad x = 2n\pi - \tfrac{1}{6}\pi.$$

33. Example II. — Solve $\sin x \tan x = -\frac{9}{20}$.

Replacing $\tan x$ by $\dfrac{\sin x}{\cos x}$ the equation becomes

$$\frac{\sin^2 x}{\cos x} = -\tfrac{9}{20},$$

which may be written

$$\frac{1 - \cos^2 x}{\cos x} = -\tfrac{9}{20} \quad \text{or} \quad \cos^2 x - \tfrac{9}{20}\cos x = 1.$$

Solving the quadratic equation, $\cos x = -\frac{4}{5}$. The angle x can be found in degrees by reference to the tables.

Problems. — Solve for x

1. $\tan x + \cot x = 2$.
2. $3 \sin x = \cos^2 x$.
3. $\sec x + \tan x = 2$.
4. $\sin x \cos x = -\frac{12}{25}$.
5. $\cos x + \cos(3x) = 0$.
6. $\tan(3x) + \tan x = 0$.

7. $\tan x = 3 \cot x$. 8. $3 \tan^2 x - 4 \sin^2 x = 1$.

9. $5 \tan^2 x - \sec^2 x = 11$. 10. $\sin (3x) + \sin (2x) = \sin x$.

Art. XV. Inverse Functions.

34. If $\sin A = a$, A is the angle whose sine is a. This is denoted by writing $A = \sin^{-1} a$, which is read A is the angle whose sine is a, A is the inverse sine of a, or A is the anti-sine of a.

For example $\sin 30° = \frac{1}{2}$, $\sin^{-1} (\frac{1}{2}) = 30°$. Notice that $\sin 30°$ has a single value $\frac{1}{2}$, but $\sin^{-1} (\frac{1}{2})$ may have any value contained in $2\pi n + 30°$ and $(2n + 1)\pi - 30°$, where n is a whole number. This fact is expressed by saying that the sine is a single-valued function, while the inverse sine is a many valued function.

In general all trigonometric functions are single valued, all inverse functions are many valued.

From equations between trigonometric functions, corresponding equations between inverse functions are readily obtained.

Example I. — From the equation

$$\sin (3A) = 3 \sin A - 4 \sin^3 A$$

derive an equation between inverse functions.

Write $\sin A = a$, whence $A = \sin^{-1} a$. The given equation becomes $\sin (3 \sin^{-1} a) = 3a - 4a^3$, whence

$$3 \sin^{-1} a = \sin^{-1} (3a - 4a^3).$$

Example II. — Prove

$$\cot^{-1} a + \cot^{-1} b = \cot^{-1} \frac{ab - 1}{a + b}.$$

Write $A = \cot^{-1} a$, $B = \cot^{-1} b$ whence $\cot A = a$, $\cot B = b$. The given equation may be written

$$\cot (\cot^{-1} a + \cot^{-1} b) = \frac{ab - 1}{a + 1}$$

which by substitution becomes

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B},$$

a true relation.

Problems. — Derive equations between inverse functions corresponding to

1. $\cos(2A) = 1 - 2\sin^2 A.$

2. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Prove the relations

3. $\sin^{-1} a - \sin^{-1} b = \sin^{-1}(a\sqrt{1-b^2} - b\sqrt{1-a^2}).$

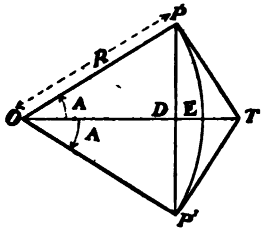
4. $3\tan^{-1} a = \tan^{-1}\dfrac{3a - a^3}{1 - 3a^2}.$

5. $\tan^{-1} a + \cos^{-1}\dfrac{1}{a} = \sin^{-1}\dfrac{a + \sqrt{a^2 - 1}}{a\sqrt{a^2 + 1}}.$

6. $2\sin^{-1} a = \tan^{-1}\dfrac{2a\sqrt{1 - a^2}}{1 - 2a^2}.$

Art. XVI. Limits of $\frac{\sin A}{A}$ and $\frac{\tan A}{A}$ when A Approaches Zero.

35. Construct an angle EOP, less than $\frac{\pi}{2}$, whose circular measure is A. Construct a second angle EOP' equal to A.



Draw the circular arc PEP' with center O, radius R. Draw the chord PP' and tangents to the arc at P and P' forming the broken line PTP'. Assuming as axiomatic that the arc is greater than the chord and less than the broken line, chord $PDP' <$ arc $PEP' <$ broken line PTP'. Dividing through by $2R$,

$$\frac{PD}{R} < \frac{PE}{R} < \frac{PT}{R},$$

that is $\sin A < A < \tan A$, hence

$$(a) \qquad \frac{1}{\sin A} > \frac{1}{A} > \frac{1}{\tan A}.$$

Multiplying through by $\sin A$,

$$1 > \frac{\sin A}{A} > \cos A.$$

If A approaches zero, $\cos A$ approaches 1. Hence the limit of $\frac{\sin A}{A}$ lies between two numbers one equal to 1, the other approaching 1, and the limit of $\frac{\sin A}{A}$ must be 1 when A approaches zero.

Multiplying (a) by $\tan A$

$$\frac{1}{\cos A} < \frac{\tan A}{A} < 1.$$

Since $\cos A$ approaches 1 when A approaches 0, $\frac{\tan A}{A}$ must approach 1 when A approaches zero.

In the following problems take $\pi = 3.1416$.

Problems. — 1. Show that the circular measures, the sines and the tangents of angles from 0° to 6° agree to three decimal places.

2. Find approximately the angle subtended by a man 6 feet high at a distance of half a mile.

3. Find approximately the distance from the eye at which a coin one inch in diameter will just hide the moon, the moon's angular diameter being $31' 5''$.

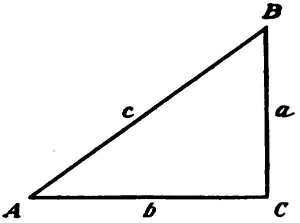
4. If the radius of the earth is 4000 miles, and if this radius as seen from the sun is $8''.8$, find the sun's distance from the earth.

PART II.

SOLUTION OF PLANE TRIANGLES.

Art. XVII. Right Triangles.

36. Let A and B be the oblique angles of a right triangle ABC, c the hypotenuse, a and b the sides opposite the angles A and B respectively.



B is a point in the terminal side of A, c the distance from B to the vertex of A, b the projection of this distance on the initial side of A, a the distance from B to the initial side of A. Hence

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}.$$

In like manner it is shown that

$$\sin B = \frac{b}{c}, \quad \cos B = \frac{a}{c}, \quad \tan B = \frac{b}{a}.$$

In words, in a right triangle

The sine of an oblique angle is the side opposite the oblique angle divided by the hypotenuse.

The cosine is the side adjacent divided by the hypotenuse.

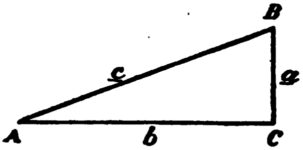
The tangent is the side opposite divided by the side adjacent.

A right triangle is determined when in addition to the right angle two parts, one of which is a side, are given. The above formulas are sufficient to compute the other parts. This computation of other parts is called solving the triangle.

The results of the computation should be checked by the formula $b^2 = c^2 - a^2$ which is not used in the computation.

37. Example I. —Solve the right triangle in which

$$c = 8.7982,\ a = 3.1292,\ C = 90^\circ.$$



The formulas for computing the parts A, B, b are

$$\sin A = \frac{a}{c},\quad B = 90^\circ - A,\quad b = c\cos A.$$

The check formula is $b^2 = c^2 - a^2 = (c + a)(c - a)$.

Apply the rules for computation by logarithms, namely,

The logarithm of the product equals the sum of the logarithms of the factors.

The logarithm of the quotient equals the logarithm of the dividend minus the logarithm of the divisor.

The above formulas become

$$\log \sin A = \log a - \log c,$$
$$\log b = \log c + \log \cos A,$$

and the check formula becomes

$$2 \log b = \log (c + a) + \log (c - a).$$

Arrange the scheme of computation as follows before referring to the tables :

$$\begin{array}{rl@{\qquad\qquad}rl}
\log a = & & \log c = & \\
- \log c = & \underline{\qquad\quad} & + \log \cos A = & \underline{\qquad\quad} \\
\log \sin A = & & \log b = & \\
A = & & b = &
\end{array}$$

Check

$$\begin{array}{rl}
\log (c + a) = & \\
+ \log (c - a) = & \underline{\qquad\quad} \\
2 \log b = & \\
\log b = &
\end{array}$$

Now take the logarithms of a and c out of the tables and A becomes known. Look up $\log \cos A$ and b becomes known. Then apply the check. The computation takes the form :

$$\begin{array}{rl}
\log a = & .49544 \\
- \log c = & \underline{.94439 \qquad\quad} \\
\log \sin A = & 9.55105 - 10 \\
A = & 20^\circ\, 50'\, 6'' \\
B = 90^\circ - A = & 69^\circ\, 9'\, 54''
\end{array}$$

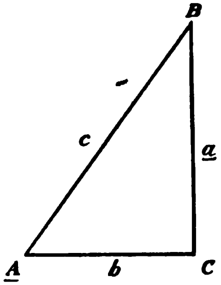
$$\begin{array}{rl}
\log c = & .94439 \\
+ \log \cos A = & \underline{9.97063 - 10} \\
\log b = & .91502 \\
b = & 8.2228
\end{array}$$

Check

$$\begin{array}{rl}
\log (c + a) = & 1.07655 \\
\log (c - a) = & \underline{.75351} \\
2 \log b = & 1.83006 \\
\log b = & .91503
\end{array}$$

Exact agreement in the check formula is not to be expected since the tables give logarithms only to the nearest unit in the fifth decimal place.

38. Example II. — Given $A = 50° 45'$, $a = 364.3$. Find b, c, B.



Formulas $B = 90° - A = 39° 15'$, $b = a \tan B$,

$$c = \frac{a}{\sin A}. \quad \textit{Check} \quad b^2 = a^2 - c^2.$$

Passing to logarithms

$$\log b = \log a + \log \tan B,$$

$$\log c = \log a - \log \sin A.$$

Check $$2 \log b = \log(c + a) + \log(c - a).$$

$$\begin{aligned} \log a &= 2.56146 \\ + \log \tan B &= \underline{9.91224 - 10} \\ \log b &= 2.47370 \\ b &= 297.64 \end{aligned}$$

$$\begin{aligned} \log a &= 2.56146 \\ - \log \sin A &= \underline{9.88896 - 10} \\ \log c &= 2.67250 \\ c &= 470.43 \end{aligned}$$

$$\begin{aligned} \log(c + a) &= 2.92155 \\ + \log(c - a) &= \underline{2.02583} \\ 2 \log b &= 4.94738 \\ \log b &= 2.47369 \end{aligned}$$

Problems. — 1. Given $c = 25.643$, $B = 37° 25' 20''$, $C = 90°$. Show that $a = 20.365$, $b = 15.583$.

2. Given $b = .356$, $B = 63° 28' 40''$, $C = 90°$. Show that $c = .3979$, $a = .1777$.

3. Given $c = 147.92$, $b = 146.39$, $C = 90°$. Show that $A = 8° 14' 52''$, $B = 81° 45' 8''$.

4. Given $a = 639.41$, $b = 417.93$, $C = 90°$. Show that $A = 56° 49' 51''$, $c = 763.87$.

5. Given $b = 93.68$, $A = 41° 6' 12''$, $C = 90°$. Show that $a = 81.73$, $c = 124.32$

Solve the triangles

6. $b = 736.3$, $c = 500$. 7. $a = 249$, $B = 29° 14'$.

8. $a = 246.32$, $b = 380.07$.

9. $b = 388.875$, $A = 27° 38' 50''$.

10. $b = 18.436$, $B = 65° 15'$.

11. $a = 7642.5$, $b = 864.7$. 12. $c = 4.8293$, $b = .31435$.

Solve the isosceles triangles

13. $a = c = 57.906$, $b = 62.736$.

14. $a = c = 500$, $A = C = 35° 25'$.

15. $a = c = 8.35$, $B = 15.47$.

Art. XVIII. Applications of Right Triangles.

1. Find the area of the regular pentagon whose side is 12 feet.

2. Find the area of the triangle whose sides are 6, 6 and 10 feet.

3. Find the area of the triangle which has two sides 30 and 50 feet, the angle included by these sides 55° 30'.

4. Show that the area of the triangle in which the sides a and b include an angle C is $\frac{1}{2}ab \sin C$.

5. Find the area of the parallelogram whose sides are 150 feet and 200 feet, and one angle 71° 25′.

6. Find the volume of the cone of revolution whose element is 10 feet, and angle at the vertex 25° 30′.

7. Find the area of the circular segment whose chord is 6 feet, if the radius of the circle is 15 feet.

8. The radii of two circles are 12.5 feet and 4.8 feet. The distance between their centers is 20 feet. Find the angle between the outer common tangents.

9. A belt is stretched around two wheels with radii 6 feet and 2 feet respectively, and with their centers 10 feet apart. Find the length of the belt.

10. On the top of a tower 125 feet high stands a pole which, at a horizontal distance of 200 feet from the foot of the tower subtends an angle of 49′. How long is the pole?

11. The Washington monument is 555 feet high. How far apart are two observers who from points due east see the top of the monument at elevations of 23° 20′ and 47° 30′?

12. From the top of a hill the angles of depression to two consecutive milestones, which are in a direction due east, are 21° 31′ and 47° 40′. How high is the hill?

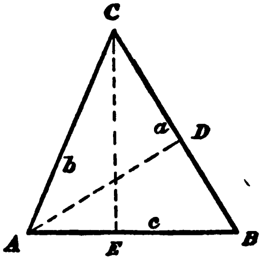
13. An inaccessible object is observed to lie due N. E. After the observer has traveled S. E. 2 miles the object lies N. N. E. Find the distance of the object from each point of observation.

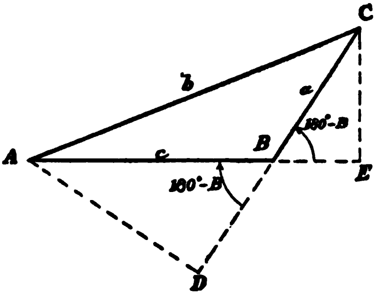
14. Suppose the earth to be a sphere with radius 4,000 miles. How far must a person be elevated above the surface in order to be able to see an object on the surface at a distance of 15 miles?

15. Find the distance travelled in an hour by reason of the earth's rotation, by an object on the surface in latitude 44° 20′.

Art. XIX. Formulas for Oblique Triangles.

39. The Sine Formula. — Let A, B, C be the three angles of an oblique triangle; a, b, c the sides opposite these angles respectively. From any vertex A draw a perpendicular AD to the opposite side. From a second vertex C draw a perpendicular CE to the side opposite that vertex.





From the right triangles ACE and BCE,

$$CE = b \sin A = a \sin B,$$

hence

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{and} \quad \frac{a}{b} = \frac{\sin A}{\sin B}.$$

From the right angles ACD and ABD,

$$AD = b \sin C = c \sin B,$$

hence

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{and} \quad \frac{b}{c} = \frac{\sin B}{\sin C}.$$

Combining the two results

$$(15) \qquad \frac{\mathbf{a}}{\mathbf{sin\ A}} = \frac{\mathbf{b}}{\mathbf{sin\ B}} = \frac{\mathbf{c}}{\mathbf{sin\ C}},$$

that is, the sides of a triangle are proportional to the sides of the angles opposite. This is known as the sine formula.

Problems. — 1. Two sides of a triangle are 10 feet and 15 feet, the angle opposite the latter is 25° 30′. Find the angle opposite the former side.

2. Two angles of a triangle are 35° 45′ and 47° 50′, the side opposite the former is 125 yards. Find the side opposite the latter angle.

3. The diagonal of a parallelogram is 100 feet and makes angles of 30° and 45° with the sides of the parallelogram. Find the lengths of the sides.

4. Prove that the ratio $\frac{a}{\sin A}$ is the diameter of the circle circumscribed about the triangle.

40. The Tangent Formula. — By composition and division there results from the proportion

$$\frac{a}{b}=\frac{\sin A}{\sin B},\qquad \frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}=$$

$$\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

Since a and b are any two sides of the triangle, in words the sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half the difference of these angles.

The formula

$$\mathbf{a}+\mathbf{b}\over\mathbf{a}-\mathbf{b}\tag{16}$$

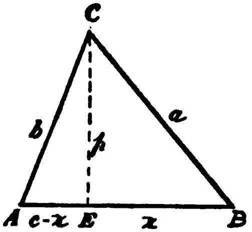
$$\frac{\mathbf{a}+\mathbf{b}}{\mathbf{a}-\mathbf{b}}=\frac{\mathbf{tan}\,\frac{1}{2}(\mathbf{A}+\mathbf{B})}{\mathbf{tan}\,\frac{1}{2}(\mathbf{A}+\mathbf{B})}\qquad(16)$$

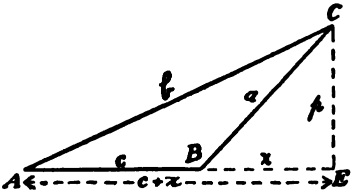
is known as the tangent formula.

Problems. — 1. Two sides of a triangle are 40 feet and 26 feet, their included angle 60°. Find the other angles of the triangle.

2. Two sides of a triangle are 100 yards and 150 yards, their included angle 100°. Find the other angles of the triangle.

41. The Cosine Formula. — Draw a perpendicular from any vertex C of the triangle to the opposite side. The perpendicular p may fall within or without the triangle.





If the perpendicular falls without the triangle

$$b^2 = p^2 + (c+x)^2 = p^2 + c^2 + x^2 + 2cx = a^2 + c^2 + 2cx.$$

From the triangle CBE,

$$x = a \cos(180° - B) = -a \cos B.$$

Hence $b^2 = a^2 + c^2 - 2ac \cos B$, that is the square of any side of a triangle equals the sum of the squares of the other two sides minus twice their product into the cosine of their included angle.

Solving for $\cos B$,

$$(17) \qquad \cos \mathbf{B} = \frac{a^2 + c^2 - b^2}{2ac}.$$

This formula, known as the cosine formula, determines an angle when the three sides are given. The formula, however, is not adapted to logarithmic computation.

Problems. — **1.** The sides of a triangle are 7, 8, 10. Find the angles.

2. The sides of a triangle are 30, 40, 60. Find the angles.

3. Two sides of a triangle are 45 and 60, their included angle 60°. Find the third side.

4. The sides of a triangle are 13, 20 and 21 inches. Find the parts into which the longest side is divided by the perpendicular from the opposite vertex.

5. Derive the sine formula from the cosine formula.

42. Formulas for sin ½B, cos ½B, tan ½B.

(*a*) Subtracting each member of $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ from unity,

$$1 - \cos B = 1 - \frac{a^2 + c^2 - b^2}{2ac} = \frac{b^2 - a^2 - c^2 + 2ac}{2ac}$$

$$= \frac{b^2 - (a - c)^2}{2ac} = \frac{\{b + (a - c)\}\{b - (a - c)\}}{2ac}$$

$$= \frac{(b + a - c)(b - a + c)}{2ac}.$$

Let $a + b + c = 2s$. Then

$$b + a - c = 2s - 2c, \quad b - a + c = 2s - 2a.$$

Since $1 - \cos B = 2 \sin^2 \tfrac{1}{2}B$, there results

$$2 \sin^2 \tfrac{1}{2}B = \frac{(2s - 2c)(2s - 2a)}{2ac},$$

whence

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(s - c)(s - a)}{ac}}.$$

(*b*) Adding each member of $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ to unity

$$1 + \cos B = 1 + \frac{a^2 + c^2 - b^2}{2ac} = \frac{a^2 + c^2 + 2ac - b^2}{2ac}$$

$$= \frac{(a + c)^2 - b^2}{2ac} = \frac{(a + b + c)(a - b + c)}{2ac}.$$

Let $a + b + c = 2s$. Then $a - b + c = 2s - 2b$.

Since $1 + \cos B = 2\cos^2 \tfrac{1}{2}B$, there results

$$2\cos^2 \tfrac{1}{2}B = \frac{2s(2s - 2b)}{2ac}, \quad \text{whence} \quad \cos \tfrac{1}{2}B = \sqrt{\frac{s(s - b)}{ac}}.$$

(*c*) Dividing $\sin \tfrac{1}{2}B$ by $\cos \tfrac{1}{2}B$,

$$\tan \tfrac{1}{2}B = \sqrt{\frac{(s - a)(s - c)}{s(s - b)}}.$$

This formula may be written

$$\tan \tfrac{1}{2}B = \sqrt{\frac{(s - a)(s - b)(s - c)}{s(s - b)^2}}$$

$$= \frac{1}{s - b}\sqrt{\frac{(s - a)(s - b)(s - b)}{s}} = \frac{k}{s - b},$$

where k stands for $\sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}}$.

These formulas for $\sin \tfrac{1}{2}B$, $\cos \tfrac{1}{2}B$, $\tan \tfrac{1}{2}B$ are adapted to logarithmic computation.

43. Direct Derivation of Formula for tan $\tfrac{1}{2}B$.

Let ABC be any triangle, r the radius of the inscribed circle. Call the distances from A to the points of tangency x, from B to the points of tangency y, from C to the points of tangency z. Then

$$x + y + z = s, \quad y + z = a, \quad x + z = b. \quad x + y = c$$

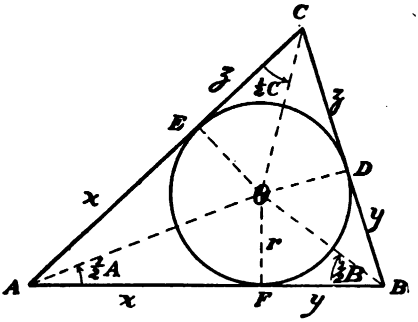
and

$$x = s - a, \quad y = s - b, \quad z = s - c.$$

Hence

$$\tan \tfrac{1}{2}A = \frac{r}{x} = \frac{r}{s - a}, \quad \tan \tfrac{1}{2}B = \frac{r}{y} = \frac{r}{s - b},$$

$$\tan \tfrac{1}{2}C = \frac{r}{z} = \frac{r}{s - c}.$$



To find r, start with $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90°$, whence $\frac{1}{2}A = 90° - (\frac{1}{2}B + \frac{1}{2}C)$. Taking the tangent of both sides,

$$\tan\tfrac{1}{2}A = \cot(\tfrac{1}{2}B + \tfrac{1}{2}C) = \frac{1}{\tan(\frac{1}{2}B + \frac{1}{2}C)} = \frac{1 - \tan\frac{1}{2}B\tan\frac{1}{2}C}{\tan\frac{1}{2}B + \tan\frac{1}{2}C}.$$

Substituting $\tan\frac{1}{2}A = \frac{r}{x}$, $\tan\frac{1}{2}B = \frac{r}{y}$, $\tan\frac{1}{2}C = \frac{r}{z}$, there results

$$\frac{r}{x} = \frac{1 - \frac{r^2}{yz}}{\frac{r}{y} + \frac{r}{z}} = \frac{yz - r^2}{ry + rz}, \quad \text{whence} \quad r^2y + r^2z = xyz - r^2x,$$

$$r^2(x + y + z) = xyz, \quad r = \sqrt{\frac{xyz}{x+y+z}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

44. Area of Triangles in Terms of Sides.

The area of the triangle is the radius of the inscribable circle multiplied by half the sum of the sides, that is area $= s.\ r$. Substituting for r the value $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$,

(18) $$\textbf{area} = \sqrt{\mathbf{s(s-a)(s-b)(s-c)}}$$

Problems. — **1.** Find the area of the triangle whose sides are 12, 15 and 20 inches.

2. Find the area of the triangle whose sides are 20, 30 and 40 feet.

Art. XX. Solution of Oblique Triangles.

45. An oblique triangle is determined when three parts, one of which is a side, are given. The formulas

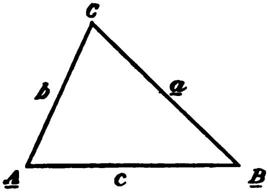
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}; \quad \frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)};$$
$$(19) \qquad \tan\tfrac{1}{2}A = \frac{k}{s-a}; \quad k = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

are sufficient to solve all oblique triangles.

When two of the three parts are opposite each other the first formula is used; when the three parts are next each other the second formula is used; when the three sides are given the third formula is used.

46. Solution When Two Angles and a Side Are Given.

Example.—Given $A = 63° 48'$, $B = 49° 25'$, $a = 825$. Find C, c, b. The formulas for solving the triangle are



$$C = 180° - (A + B) = 66° 47',$$

$$c = \frac{a \sin C}{\sin A} \quad \text{and} \quad b = \frac{a \sin B}{\sin A}.$$

Check the computation by the formula

$$\frac{c+b}{c-b} = \frac{\tan\frac{1}{2}(C+B)}{\tan\frac{1}{2}(C-B)}.$$

Applying logarithms these formulas become

$$\log c = \log a + \log \sin C - \log \sin A,$$

$$\log b = \log a + \log \sin B - \log \sin A$$

and

$$\log (c + b) - \log (c - b) =$$
$$\log \tan \tfrac{1}{2}(C + B) - \log \tan \tfrac{1}{2}(C - B).$$

Arrange the computation as follows :

$$\begin{array}{rl}
\log a = & 2.91645 \\
+ \log \sin C = & \underline{9.96333 - 10} \\
 & 12.87978 - 10 \\
- \log \sin A = & \underline{9.95292 - 10} \\
\log c = & 2.92686 \\
c = & 845
\end{array}$$

$$\begin{array}{rl}
\log a = & 2.91645 \\
+ \log \sin B = & \underline{9.88051 - 10} \\
 & 12.79696 - 10 \\
- \log \sin A = & \underline{9.95292 - 10} \\
\log b = & 2.84404 \\
b = & 698.3
\end{array}$$

Check

$$\begin{array}{rl}
\log (c + b) = & 3.18845 \\
- \log (c - b) = & \underline{2.16643} \\
 & 1.02202
\end{array}$$

$$\begin{array}{rl}
\log \tan \frac{1}{2}(C + B) = & .20590 \\
- \log \tan \frac{1}{2}(C - B) = & \underline{9.18391 - 10} \\
 & 1.02199
\end{array}$$

Problems. — **1.** Given

$$A = 57^\circ\, 23'\, 12'', \quad C = 68^\circ\, 15'\, 30'', \quad c = 832.56.$$

Show that

$$B = 54^\circ\, 21'\, 18'', \quad a = 754.98, \quad b = 728.38.$$

2. Given

$$A = 47° 19' 18'', \quad B = 78° 14' 42'', \quad a = 738.1.$$

Show that

$$C = 54° 26'; \quad b = 982.9, \quad c = 816.7.$$

3. Given

$$B = 66° 39', \quad C = 18° 27', \quad c = 1764.3.$$

Show that

$$A = 94° 54', \quad a = 5554.4, \quad b = 5118.2.$$

4. Given

$$A = 99° 55', \quad B = 45° 1', \quad a = 804.$$

Show that

$$C = 35° 4', \quad b = 577.30, \quad c = 468.93.$$

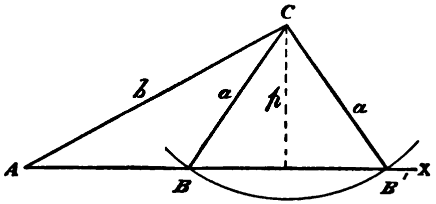
5. Given

$$B = 13° 57', \quad C = 57° 13', \quad b = 67.85.$$

Show that

$$A = 108° 50', \quad a = 266.4, \quad c = 236.6.$$

47. To Construct a Triangle having Given Two Sides and the Angle Opposite One of these Sides.—Let the given parts be a, b, A. Lay off the angle A and the side b. The vertex B must be in the line AX and in the circumference struck off from C as center with radius a. When a is less than the perpendicular p this circumference does not intersect AX, and consequently the triangle is impossible. When $a = p$, circumference and line AX have one point in common, and the triangle con-



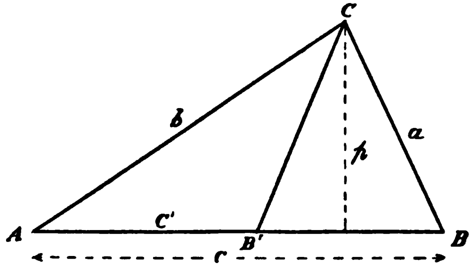
structed is a right triangle. When a is greater than p and less than b, two triangles are possible with the given parts. When $a > b$, the circumference intersects AX in two points, but only the triangle constructed with the point of intersection to the right of A contains the three given parts.

The length of the perpendicular is $p = b \sin A$, whence $\log p = \log b + \log \sin A$. When solving a triangle it is convenient to determine the number of solutions by the rule.

If $\log a < \log p$ the triangle is impossible.
If $\log a = \log p$ the triangle is a right triangle.
If $\log p < \log a < \log b$ there are two triangles.
If $a > b$ there is only one triangle.

48. Solution when Two Sides and an Angle Opposite One of these Sides are Given.

Example. — Given $b = 23.647$, $a = 14.135$, $A = 33° 17' 18''$. Find B, A, c.



The formulas are

$$\frac{b}{a} = \frac{\sin B}{\sin A} \quad \text{whence} \quad \sin B = \frac{b \sin A}{a}; \; C = 180° - (A + B);$$

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \quad \text{whence} \quad c = \frac{a \sin C}{\sin A}.$$

Use $\dfrac{b+c}{b-c} = \dfrac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)}$ to check the computation.

Arrange the computation as follows :

$$\begin{aligned} \log b &= 1.37378 \\ +\log\sin A &= \underline{9.73946-10} \\ \log p &= 11.11324-10 \\ -\log a &= \underline{1.15029} \\ \log\sin B &= 9.96295-10 \\ B &= 66^\circ\, 40' \\ B' &= 113^\circ\, 20' \\ C &= 80^\circ\, 2'\, 42'' \\ C' &= 33^\circ\, 22'\, 42'' \end{aligned}$$

Since

$\log b > \log a > \log p$

there are two solutions.

$$\begin{aligned} \log a &= 1.15029 \\ +\log\sin C &= \underline{9.99341-10} \\ & \ 11.14370-10 \\ -\log\sin A &= \underline{9.73946-10} \\ \log c &= 1.40424 \\ c &= 25.365 \end{aligned}$$

$$\begin{aligned} \log a &= 1.15029 \\ +\log\sin C' &= \underline{9.74047-10} \\ & \ 10.89076-10 \\ -\log\sin A &= 9.73945-10 \\ \log c' &= 1.15131 \\ c' &= 14.168 \end{aligned}$$

Check

$$\begin{aligned} \log(b+c) &= 1.57769 \\ +\log\tan\tfrac{1}{2}(B'-C') &= \underline{9.92347-10} \\ & \ 1.50116 \\ \log(b-c') &= .97676 \\ +\log\tan\tfrac{1}{2}(B'+C') &= \underline{.52440} \\ & \ 1.50116 \end{aligned}$$

Problems.—1. Given $A = 40^\circ$, $a = 350$, $c = 400$. Show that

$$C = 47^\circ\, 16',\ 28'', \quad B = 92^\circ\, 43'\, 32'', \quad b = 543.89,$$

$C = 132° 43' 32''$, $B' = 7° 16' 28''$, $b' = 68.94$.

2. Given $b = 180$, $c = 100$, $B = 127° 33'$.

Show that

$$C = 26° 7' 59'', \quad A = 26° 19' 1'', \quad a = 100.65.$$

3. Given $a = 600$, $b = 250$, $B = 42° 12'$.
Show that a triangle cannot be constructed.

4. Given $a = 4.254$, $c = 4.536$, $C = 37° 9'$.

Show that

$$B = 108° 21' 12'', \quad A = 34° 29' 48'', \quad b = 7.129.$$

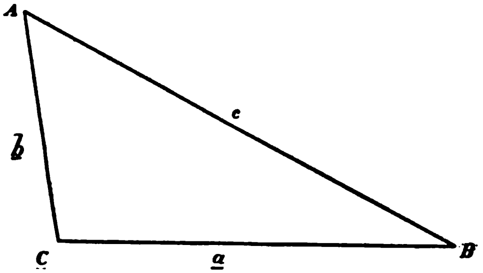
5. Given $A = 38° 54' 42''$, $a = 625.89$, $b = 739.47$.

Show that

$$B = 47° 54' 44'', \quad C = 93° 10' 34'', \quad c = 994.9.$$
$$B' = 132° 5' 16'', \quad C' = 9° 0' 2'', \quad c' = 155.89.$$

49. Solution when Two Sides and the Included Angle are Given.

Example. — Given $a = 1686$, $b = 960$, $C = 128° 44'$.
Find A, B, c.



The formulas are $A + B = 180° - C = 51° 16'$,

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

whence

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B);$$

$$\frac{c}{a} = \frac{\sin C}{\sin A} \quad \text{whence} \quad c = \frac{a \sin C}{\sin A}.$$

Use $\frac{b}{c} = \frac{\sin B}{\sin C}$ to check the computation.

$$\begin{aligned}
\log (a - b) &= 2.86094 \\
+ \log \tan \tfrac{1}{2}(A + B) &= \underline{9.68754 - 10} \\
& 12.54848 - 10 \\
- \log (a + b) &= \underline{3.42259} \\
\log \tan \tfrac{1}{2}(A - B) &= 9.12589 - 10 \\
\tfrac{1}{2}(A - B) &= 7^\circ\, 36'\, 40''. \\
\tfrac{1}{2}(A + B) &= 25^\circ\, 58'. \\
B &= 33^\circ\, 34'\, 40''. \\
B &= 18^\circ\, 21'\, 20''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 3.22686 \\
+ \log \sin C &= \underline{9.89614 - 10} \\
& 13.12300 - 10 \\
- \log \sin A &= \underline{9.74277} \\
\log C &= 3.38023 \\
C &= 2400.1
\end{aligned}$$

$$\begin{aligned}
&\textit{Check} \quad \log b = 2.98227 & \log c &= 3.38023 \\
&+ \log \sin C = \underline{9.89614 - 10} & + \log \sin B &= \underline{9.49819 - 10} \\
& 2.87841 & & 2.87842
\end{aligned}$$

Problems. — 1. Given $b = 472$, $c = 324$, $A = 78^\circ\, 40'$, Show that $B = 63^\circ\, 27'\, 1''$, $C = 37^\circ\, 52'\, 59''$, $a = 517.355$.

2. Given $a = 49.387$, $b = 41.414$, $C = 109^\circ\, 38'\, 36''$. Show that $A = 38^\circ\, 43'\, 12''$, $B = 31^\circ\, 38'\, 12''$, $c = 74.36$.

3. Given $a = 101.47$, $c = 99.367$, $B = 47^\circ\, 48'\, 12''$. Show that $A = 67^\circ\, 27'\, 6''$, $C = 64^\circ\, 44'\, 42''$, $b = 81.396$.

4. Given $b = 456.12$, $c = 296.86$, $A = 74° 20'$.
Show that $B = 68° 25' 18''$, $C = 37° 14' 42''$.

5. Given $b = 99.685$, $c = 312.375$, $A = 130° 9' 24''$.
Show that $B = 11° 26''$, $C = 38° 24' 36''$, $a = 384.3$.

50. Solution When the Three Sides Are Given.

Example. — Given $a = 3459$, $b = 4209$, $c = 6030$.
Find A, B, C.

The formulas are

$$\tan \tfrac{1}{2}A = \frac{k}{s-a}, \quad \tan \tfrac{1}{2}B = \frac{k}{s-b}, \quad \tan \tfrac{1}{2}C = \frac{k}{s-c},$$

$$k = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Check the computation by the formula

$$\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90°.$$

$a =$	3459		
$b =$	4209	$\log(s-a) =$	3.53020
$c =$	6030	$+\log(s-b) =$	3.42160
$2s =$	13698	$+\log(s-c) =$	2.91328
$s =$	6849		9.86508
$s-a =$	3390	$-\log s =$	3.83563
$s-b =$	2640	$2\log k =$	6.02945
$s-c =$	819	$\log k =$	3.01472
$\log \tan \tfrac{1}{2}A =$	9.48452	$\tfrac{1}{2}A =$	$16° 58' 13''$
$\log \tan \tfrac{1}{2}B =$	9.59312	$\tfrac{1}{2}B =$	$21° 23' 52''$
$\log \tan \tfrac{1}{2}C =$	.10144	$\tfrac{1}{2}C =$	$51° 37' 53''$

Check $\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 89° 59' 58''$.

Problems. — 1. Given

$$a = 32.456, \quad b = 41.724, \quad c = 53.987.$$

Show that

$$\tfrac{1}{2}A = 18^\circ\, 27'\, 24'', \quad \tfrac{1}{2}B = 25^\circ\, 16'\, 18'', \quad \tfrac{1}{2}C = 46^\circ\, 16'\, 24''.$$

2. Given

$$a = .785, \quad b = .85, \quad c = .633.$$

Show that

$$\tfrac{1}{2}A = 30^\circ\, 56'\, 49'', \quad \tfrac{1}{2}B = 36^\circ\, 23'\, 2'', \quad \tfrac{1}{2}C = 22^\circ\, 40'\, 10''.$$

3. Given

$$a = 63.89, \quad b = 138.24, \quad c = 121.15.$$

Show that

$$\tfrac{1}{2}A = 13^\circ\, 45'\, 35'', \quad \tfrac{1}{2}B = 45^\circ\, 39', \quad \tfrac{1}{2}C = 30^\circ\, 35'\, 27''.$$

4. Given

$$a = 25.17, \quad b = 34.06, \quad c = 22.17.$$

Show that

$$\tfrac{1}{2}A = 23^\circ\, 48'\, 28'', \quad \tfrac{1}{2}B = 45^\circ\, 54', \quad \tfrac{1}{2}C = 20^\circ\, 17'\, 35''.$$

5. Given

$$a = 500, \quad b = 403.7, \quad c = 395.75.$$

Show that

$$A = 77^\circ\, 25'\, 12'', \quad B = 52^\circ\, 0'\, 3'', \quad C = 50^\circ\, 34'\, 45''.$$

Art. XXI. Applications.

1. The sides of a four sided field are $AB = 155$ rods, $BC = 236$ rods, $CD = 252$ rods, $DA = 105$ rods. The diagonal $AC = 311$ rods. Find the area of the field.

2. Two trains start from the same point at the same time and move along straight railways which intersect at an angle 74° 30′. The rates of the trains are 30 and 45 miles per hour respectively. How far apart are they at the end of 45 minutes.

3. To find the height of a hill above a horizontal plane a base line 975 yards long is measured in the horizontal plane and lying in the vertical plane through the top of the hill. From the ends

of the line the angles of elevation to the top are 15° 36′ and 27° 29′. Find the height of the hill.

4. A base line 850 feet long is measured along the straight bank of a river. At the ends of this line the lines of sight to an object on the opposite bank make angles of 63° 40′ and 37° 15′ with the base line. Find the width of the river.

5. The distances of a station from two objects situated on opposite sides of a hill are 1128 yards and 936 yards. The angle subtended at the station by the distance between the two objects is 64° 28′. Find the distance between the objects.

6. To determine the height of a mountain a north and south base line 1000 yards long is measured. From one end of the base line the top of the mountain bears E. 10° N., and is at an altitude of 13° 14′; from the other end it bears E. 46° 30′ N. Find the height of the mountain. Ans., 272.14 yards.

7. A person goes 84 yards up a slope of 2 feet in 7 from the edge of a river and observes that the angle of depression of the edge of the water on the opposite is 2° 15′. Find the width of the river. Ans., 530.53 yards.

8. To determine the length of a pond a base line 100 yards long is measured. At one end of the base line the angles included by the base line and the lines of sight to the extremities of the pond are 32° and 98°. At the other end of the base line these angles are 37° and 118°. Find the length of the pond. Ans., 161.87 yards.

9. The zenith distance of the moon at any point of the earth's surface is the angle between the radius of the earth at the point and the line connecting this point with the center of the moon. If the radius of the earth is 3956.2 miles and two observers 92° 14′ apart at the same instant find the zenith distance of the moon to be 44° 54′ 21″ and 48° 42′ 57″, show that the distance

from the center of the earth to the moon is at that instant 237,954 miles.

10. A spectator observes the explosion of a meteor, due south of him, at an elevation of 28° 45′. To another spectator, 11 miles S. S. W. of the former, it appears at the same instant to have an altitude of 42° 15′ 30″. Show that the height of the meteor must be either 4.33 miles or 13.21 miles.

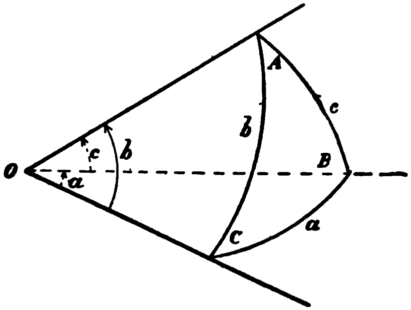
PART III.

SPHERICAL TRIGONOMETRY.

Art. XXII. Formulas of Spherical Trigonometry.

51. Introduction. — Spherical trigonometry discusses the relations between the trigonometric functions of the three vertex angles and the three edge angles of a triedral angle. These six angles are called the six parts of the triedral angle.

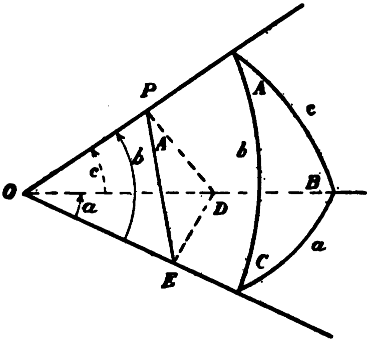
If the surface of a sphere is described with the vertex of the triedral angle as center, the planes of the vertex angles of the triedral angle intersect the spherical surface in three arcs of great circles which bound a spherical triangle.



The edge angles of the triedral angle are equal to the angles of the spherical triangle ; the vertex angles of the triedral angle have the same measure as the sides of the spherical triangle. Hence the discussion of the relations between the parts of the triedral angle is equivalent to the discussion of the relations between the sides and angles of the spherical triangle. This explains the use of the term spherical trigonometry. Frequently it is more convenient to speak of the relations between the parts of the spherical triangle than of the relations between the parts of the triedral angle.

The properties of spherical triangles learned in the geometry of three dimensions are applicable in spherical trigonometry. It will be assumed that each of the parts of the triedral angle or of the spherical triangle is less than 180°.

52. The Fundamental Formulas. — Let O be any triedral angle, P any point in one of its edges. Through P pass a plane perpendicular to the edge OA. This plane intersects the triedral angle in the triangle PDE, and the angle DPE equals the edge angle OA or the spherical angle A.



Applying the cosine formula (17) in the triangle ODE,

$$\cos a = \frac{\overline{OE}^2 + \overline{OD}^2 - \overline{DE}^2}{2OE \cdot OD}$$

and in the triangle PDE,

$$\overline{DE}^2 = \overline{PE}^2 + \overline{PD}^2 - 2PE \cdot PD \cos A.$$

Substituting this value of $\overline{DE}^2$ in $\cos a$,

$$\cos a = \frac{\overline{OE}^2 + \overline{OD}^2 - \overline{PE}^2 - \overline{PD}^2 + 2PE \cdot PD \cos A}{2OE \cdot OD}.$$

From the right triangles OPD and OPE,

$$\overline{OE}^2 - \overline{PE}^2 = \overline{OP}^2 \quad \text{and} \quad \overline{OD}^2 - \overline{PD}^2 = \overline{OP}^2.$$

Hence

$$\cos a = \frac{\overline{OP}^2 + PE \cdot PD \cos A}{OE \cdot OD} = \frac{OP}{OE} \cdot \frac{OP}{OD} + \frac{PE}{OE} \cdot \frac{PD}{OD} \cos A.$$

Finally (a) $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

This is the cosine formula of spherical trigonometry.

The triedral angle in the figure is so drawn that the perpendicular plane through P intersects the three edges of the triedral angle. However if this plane intersects one or more edges produced the same argument establishes formula (a). Formula (a) is therefore of general application.

If planes are drawn perpendicular to the edges OB and OC two analogous formulas are established.

The three formulas,

$$\begin{aligned} &\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos A,} \\ (19) \qquad &\mathbf{\cos b = \cos a \cos c + \sin a \sin c \cos B,} \\ &\mathbf{\cos c = \cos a \cos b + \sin a \sin b \cos C,} \end{aligned}$$

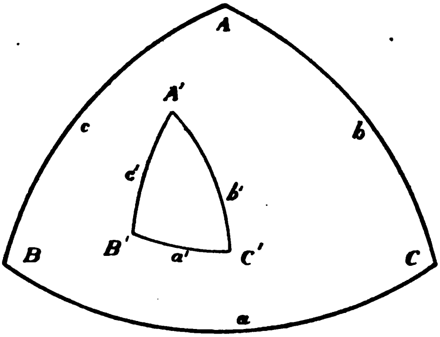
express relations between the six parts of a triedral angle. Hence if any three parts are given the other three parts may be found by means of these formulas.

All the formulas of spherical trigonometry may be derived from these three formulas, which are therefore called the fundamental formulas of spherical trigonometry.

Observe that the second formula may be obtained from the first by interchanging a and b, A and B; the third from the second by interchanging b and c, B and C. It follows that the same interchange of letters in any formula derived from the fundamental formulas must give a true formula.

53. Applying Formulas to the Polar Triangle.

If ABC and $A'B'C'$ are two polar spherical triangles, by geometry $A = 180° - a'$, $B = 180° - b'$, $C = 180° - c'$, $a = 180° - A'$, $b = 180° - B'$, $c = 180° - C'$. Substituting



in the formulas (19) and dropping accents as the results are general,

$$
(20)\qquad
\begin{aligned}
\cos \mathbf{A} &= -\cos \mathbf{B} \cos \mathbf{C} + \sin \mathbf{B} \sin \mathbf{C} \cos \mathbf{a},\\
\cos \mathbf{B} &= -\cos \mathbf{A} \cos \mathbf{C} + \sin \mathbf{A} \sin \mathbf{C} \cos \mathbf{b},\\
\cos \mathbf{C} &= -\cos \mathbf{A} \cos \mathbf{B} + \sin \mathbf{A} \sin \mathbf{B} \cos \mathbf{c}.
\end{aligned}
$$

This process of passing from formulas (19) to formulas (20) is called applying formulas (19) to the polar triangle.

54. The Sine Formula. — From formulas (19)

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Squaring, then subtracting each side from unity, and remembering that $1 - \cos^2 A = \sin^2 A$, we have

$$\sin^2 A = \frac{\sin^2 b \sin^2 c - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c}{\sin^2 b \sin^2 c}.$$

Substituting in the numerator

$$\sin^2 b = 1 - \cos^2 b, \quad \sin^2 c = 1 - \cos^2 c,$$

and dividing through by $\sin^2 a$,

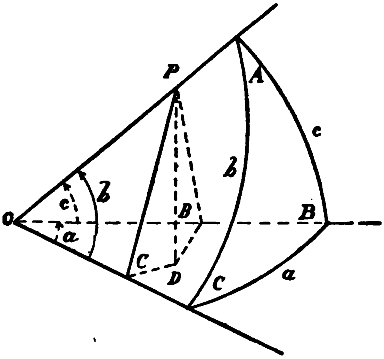
$$(a) \quad \frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

By interchanging letters in (a) $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ are found to have the same value as $\frac{\sin^2 A}{\sin^2 a}$. Hence equating and extracting the square root,

$$(21) \qquad \frac{\sin \mathrm{A}}{\sin \mathrm{a}} = \frac{\sin \mathrm{B}}{\sin \mathrm{b}} = \frac{\sin \mathrm{C}}{\sin \mathrm{c}}.$$

This is the sine formula of spherical trigonometry.

The sine formula is readily proved from the figure. From any point P in the edge OA draw PD perpendicular to the opposite face OBC. Through PD pass a plane perpendicular to the edge OB, another plane perpendicular to the edge OC.



From the right triangles *PDB* and *PDC*,

$$\sin B = \frac{PD}{PB}, \quad \sin C = \frac{PD}{PC}.$$

Dividing,

$$\frac{\sin B}{\sin C} = \frac{PC}{PB}.$$

From the right triangles *OCP* and *OBP*,

$$\sin b = \frac{PC}{OP}, \quad \sin c = \frac{PB}{OP}.$$

Dividing,

$$\frac{\sin b}{\sin c} = \frac{PC}{PB}.$$

Hence

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

55. Relations between Five Parts. — From the formulas

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

and

$$\cos b = \cos a \cos c + \sin a \sin c \cos B,$$

$$\sin a \sin b \cos C = \cos c - \cos^2 a \cos c - \sin a \sin c \cos a \cos B$$

$$= \cos c \sin^2 a - \sin a \sin c \cos a \cos B.$$

Dividing by sin *a*,

(22) $$\mathbf{\sin b \cos C = \sin a \cos c - \sin c \cos a \cos B.}$$

By interchanging letters five additional relations analogous to (22) are obtained.

By applying (22) to the polar triangle,

(23) $$\mathbf{\sin B \cos c = \sin A \cos C + \sin C \cos A \cos b.}$$

By interchanging letters five additional relations analogous to (23) are obtained.

Dividing (22) member for member by

$$\sin b \sin C = \sin c \sin B, \cdots$$

there results

$$\cot C = \frac{\sin a \cot c - \cos a \cos B}{\sin B},$$

whence

$$\text{(24)} \qquad \sin a \cot c = \sin B \cot C + \cos a \cos B.$$

Five additional relations analogous to (24) are obtained by interchanging letters in (24).

By applying (24) to the polar triangle and then interchanging letters six more relations are found. One of these six relations is

$$\text{(25)} \qquad \sin C \cot A = \cot a \sin b - \cos b \cos C.$$

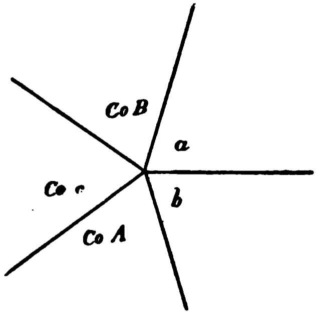
Art. XXIII. Solution of Right Spherical Triangles.

56. Formulas. — A right spherical triangle becomes known when in addition to the right angle any two of the remaining five parts are given. Hence the set of formulas containing all combinations by twos of the five parts will be sufficient to solve right spherical triangles. The number of combinations of five things by twos is ten. The following ten formulas for solving right spherical triangles are found by placing $C = 90°$ in the formulas (19), (20), (21), (22), (23), (24), (25).

$$\text{(26)} \qquad \begin{aligned} \cos c &= \cos a \cos b & \qquad \cos c &= \cot A \cot B \\ \cos A &= \sin B \cos a & \cos A &= \tan b \cot c \\ \cos B &= \sin A \cos b & \cos B &= \tan a \cot c \\ \sin a &= \sin A \sin c & \sin a &= \tan b \cot B \\ \sin b &= \sin B \sin c & \sin b &= \tan a \cot A \end{aligned}$$

57. Napier's Rules. — These ten formulas are readily remembered by the aid of two rules discovered by Napier, a Scotch mathematician.

The two sides about the right angle and the complements of the other three parts are called the five circular parts of the right spherical triangle. Arrange the circular parts as in the figure.



Select any one of the five circular parts as the middle part. Notice that two parts are adjacent to the middle part. The remaining two parts are said to be opposite to the middle part. Notice also that if any three parts are given it is always possible to select one of them as a middle part so that the other two parts are either both adjacent to it or both opposite to it.

Napier's rules are

I. The sine of the middle part is the product of the cosines of the opposite parts.

II. The sine of the middle part is the product of the tangents of the adjacent parts.

58. Geometric Proof of Formulas. — Seven of formulas (26) may be proved directly from the figure as follows. The remaining three may be derived from these seven.

Through any point P of the edge OB of the triedal angle pass a plane perpendicular to the edge OB. The angles OPD, OPE and PDE are right angles and the angle DPE equals the edge angle OB or the spherical angle B.

In the right triangle *PDE*, Fig. *A*,

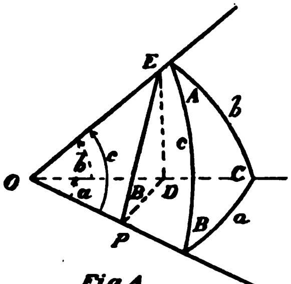


Fig. A

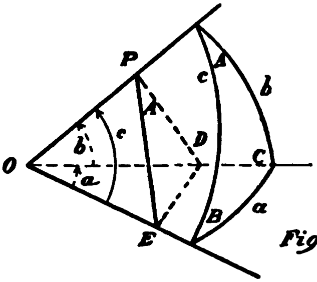


Fig. B

$$\sin B = \frac{DE}{PE} = \frac{\frac{DE}{OE}}{\frac{PE}{OE}} = \frac{\sin b}{\sin c}. \quad \text{Hence} \quad \sin b = \sin B \sin c.$$

$$\cos B = \frac{PD}{PE} = \frac{\frac{PD}{OP}}{\frac{PE}{OP}} = \frac{\tan a}{\tan c}. \quad \text{Hence} \quad \cos B = \tan a \cot c.$$

$$\cot B = \frac{PD}{ED} = \frac{\frac{PD}{OD}}{\frac{ED}{OD}} = \frac{\sin a}{\tan b}. \quad \text{Hence} \quad \sin a = \tan b \cot B.$$

In like manner from Fig. *B*,

$\sin a = \sin A \sin c$, $\cos A = \tan b \cot c$, $\sin b = \tan a \cot A$.

In the right triangle *OPD*, Fig. *B*,

$$\cos a = \frac{OP}{OD} = \frac{\frac{OP}{OE}}{\frac{OD}{OE}} = \frac{\cos c}{\cos b}. \quad \text{Hence} \quad \cos c = \cos a \cos b.$$

Multiplying

$$\sin B = \frac{\sin b}{\sin c} \quad \text{by} \quad \cos a = \frac{\cos c}{\cos b},$$

$\sin B \cos a = \tan b \cot c = \cos A$. In like manner the relation $\sin A \cos b = \cos B$ is proved.

Multiplying $\cot A = \frac{\sin b}{\tan a}$ by $\cot B = \frac{\sin a}{\tan b}$, the relation $\cot A \cot B = \cos a \cos b = \cos c$ is proved.

59. Rules for Quadrants. — Since the parts of a right spherical triangle may end either in the first or in the second quadrant it is necessary to determine in what quadrant a computed part must end.

If the computed part is found in terms of the cosine, tangent, or cotangent the algebraic sign of the function determines in what quadrant the part ends.

If the computed part is found in terms of the sine, the quadrant is determined as follows:

In the formula $\cos c = \cos a \cos b$, if c is less than 90° $\cos c$ is positive and therefore $\cos a$ and $\cos b$ must have like signs. If $\cos a$ and $\cos b$ are positive, a and b both end in the first quadrant; if $\cos a$ and $\cos b$ are negative, a and b both end in the second quadrant. If c is greater than 90°, $\cos c$ is negative and $\cos a$ and $\cos b$ have unlike signs. Hence a and b must end in different quadrants.

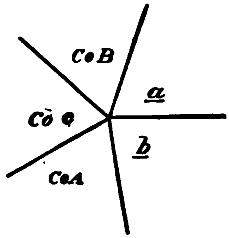
In the formula $\sin a = \tan b \cot B$, $\sin a$ is positive. Hence $\tan b$ and $\cot B$ must either both be positive or both negative, and b and B must both be less than 90° or both greater than 90°.

These results are formulated in the rules:

> In a right spherical triangle if the hypotenuse is less than 90° the sides about the right angle end in the same quadrant, if the hypotenuse is greater than 90° the sides about the right angle end in different quadrants.

An oblique angle and the side opposite end in the same quadrant.

60. Example I.—Given $a = 132° 6'$, $b = 77° 51'$. Find A, B and c.



Consider the circular parts a, b and co A. Of these parts b is the middle part, a and co A are adjacent.

Applying Napier's rules, $\overset{+}{\sin}\, b = \overset{-}{\tan}\, a\, \underline{\overset{-}{\cot}\, A}$, where the algebraic signs of the functions are placed above the trigonometric functions and the function to be found is underscored. From this formula cot A becomes known and the minus sign shows that A is the supplement of the angle to be taken out of the tables.

Next consider the circular parts a, b and co B. Of these parts a is the middle part and the parts b and co B are adjacent. Hence by Napier's rules $\overset{+}{\sin}\, a = \overset{+}{\tan}\, b\, \underline{\overset{+}{\cot}\, B}$, and cot B becomes known and B is less than 90°.

Lastly consider a, b and co c. Of these parts co c is the middle part and a and b are opposite. Hence by Napier's rules $\underline{\overset{-}{\cos}\, c} = \overset{-}{\cos}\, a\, \overset{+}{\cos}\, b$ and cos c becomes known and c is the supplement of the angle taken out of the tables.

To check the computation, consider the parts found, co c, co A and co B, of which co c is the middle part, the other two parts opposite. Napier's rule gives $\cos c = \cot A \cot B$.

Check the quadrants in which the parts found end by the rule for quadrants.

Arrange the scheme of computation as follows:

$$\begin{array}{rl@{\qquad}rl}
\log\sin b = & & \log\sin a = & \\
-\log\tan a = & \underline{\qquad\qquad} & -\log\tan b = & \underline{\qquad\qquad} \\
\log\cot A = & & \log\cot B = & \\
A = 180^\circ - & \qquad = & B = &
\end{array}$$

Check

$$\begin{array}{rl@{\qquad}rl}
\log\cos a = & & \log\cot A = & \\
+\log\cos b = & \underline{\qquad\qquad} & +\log\cot B = & \underline{\qquad\qquad} \\
\log\cos c = & & \log\cos c = & \\
c = 180^\circ - & \qquad = & &
\end{array}$$

Now fill in the numerical work. One reference to the tables gives sin b, tan b and cos b; a second reference gives sin a, tan a and cos a. The computation appears as follows:

$$\begin{array}{rl@{\qquad}rl}
\log\sin b = & 9.99016 & \log\sin a = & 9.87039 \\
-\log\tan a = & \underline{.04404} & -\log\tan b = & \underline{.66697} \\
\log\cot A = & 9.94612 & \log\cot B = & 9.20342 \\
A = & 180^\circ - 48^\circ\, 32''\, 42 & B = & 80^\circ\, 55'\, 24'' \\
= & 131^\circ\, 27'\, 18'' & &
\end{array}$$

Check

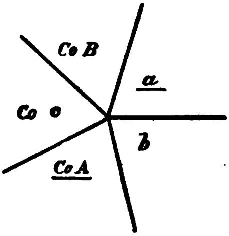
$$\begin{array}{rl@{\qquad}rl}
\log\cos a = & 9.82635 & \log\cot A = & 9.94612 \\
+\log\cos b = & \underline{9.32319} & +\log\cot B = & \underline{9.20342} \\
\log\cos c = & 9.14954 & & 9.14954 \\
c = & 180^\circ - 81^\circ\, 53'\, 18'' & \log c = & 9.14954 \\
= & 98^\circ\, 6'\, 42'' & &
\end{array}$$

The results agree with the rules for quadrants.

61. Example II.—Given $b = 126^\circ\, 38'\, 12''$, $A = 67^\circ\, 3'\, 42''$. Find B, a, c.

By Napier's rules,

$$\underline{\overset{-}{\cos}\, A} = \overset{-}{\cos}\, b\, \overset{+}{\sin}\, A. \quad \text{Hence} \quad B > 90^\circ.$$



$$\overset{+}{\sin}\, b = \underline{\overset{+}{\tan}\, a}\, \overset{-}{\cot} A. \quad \text{Hence} \quad a < 90°.$$

$$\overset{+}{\cos} A = \overset{-}{\tan}\, b\, \underline{\overset{-}{\cot}\, c}. \quad \text{Hence} \quad c > 90°.$$

Check $\cos B = \tan a \cot c.$

The work is as follows:

$$\begin{aligned} \log \cos b &= 9.77578 \\ + \log \sin A &= \underline{9.96422} \\ \log \cos B &= 9.74000 \\ B &= 180° - 56° 39' 51'' \\ &= 123° 20' 9'' \end{aligned} \qquad \begin{aligned} \log \sin b &= 9.90441 \\ - \log \cot A &= \underline{9.62656} \\ \log \tan a &= .27785 \\ a &= 62° 11' 32'' \end{aligned}$$

Check

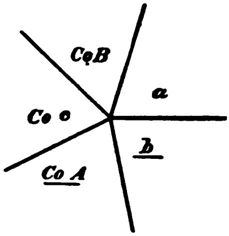
$$\begin{aligned} \log \cos A &= 9.59078 \\ - \log \tan b &= \underline{.12863} \\ \log \cot c &= 9.46215 \\ c &= 180° - 73° 50' 12'' \\ &= 106° 9' 48'' \end{aligned} \qquad \begin{aligned} \log \tan a &= .27785 \\ + \log \cot c &= \underline{9.46215} \\ & 9.74000 \\ \log \cos B &= 9.74000 \end{aligned}$$

62. Example III.—Given $A = 100°$, $a = 112°$. Find B, b and c.

By Napier's rules, $\cos A = \cos a\, \underline{\sin B},$

$$\underline{\sin b} = \tan a \cot A,$$

$$\sin a = \underline{\sin c} \sin A.$$



From the formula last written, $\sin c = \dfrac{\sin a}{\sin A}$. Hence the triangle is impossible if the sine of the side is greater than the sine of the opposite angle.

The solution is as follows :

$$\begin{aligned}
\log\cos A &= 9.23967\\
-\log\cos a &= \underline{9.57358}\\
\log\sin B &= 9.66609\\
B &= 27^\circ\,36'\,59''\\
\text{or}\quad & 152^\circ\,23'\,1''
\end{aligned}
\qquad\qquad
\begin{aligned}
\log\tan a &= .39359\\
+\log\cot A &= \underline{9.24632}\\
\log\sin b &= 9.63991\\
b &= 25^\circ\,52'\,33''\\
\text{or}\quad & 154^\circ\,7'\,27''
\end{aligned}$$

Check

$$\begin{aligned}
\log\sin a &= 9.96717\\
-\log\sin A &= \underline{9.99335}\\
\log\sin c &= 9.97382\\
c &= 70^\circ\,18'\,10''\\
\text{or}\quad & 109^\circ\,41'\,50''
\end{aligned}
\qquad\qquad
\begin{aligned}
\log\sin c &= 9.97382\\
+\log\sin B &= \underline{9.66609}\\
& 9.63991\\
\log\sin b &= 9.63991
\end{aligned}$$

Arranging the parts of the two triangles according to the law of quadrants, the two solutions are

$$\begin{aligned}
\text{I.}\quad A &= 100^\circ.\\
a &= 112^\circ.\\
c &= 70^\circ\,18'\,10''.\\
b &= 154^\circ\,7'\,27''.\\
B &= 152^\circ\,23'\,1''.
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{II.}\quad A &= 100^\circ.\\
a &= 112^\circ.\\
c &= 109^\circ\,41'\,40''.\\
b &= 25^\circ\,52'\,33''.\\
B &= 27^\circ\,36'\,59''.
\end{aligned}$$

Problems. — Solve the following right spherical triangles.

1. $a = 50^\circ 45'$, $c = 110^\circ 30'$, $C = 90^\circ$.

2. $a = 151^\circ$, $A = 125^\circ 31'$, $C = 90^\circ$.

3. $a = 118^\circ$, $B = 12^\circ 19'$.

4. $A = 113^\circ 9' 12''$, $B = 130^\circ 18' 18''$, $C = 90^\circ$.

5. $b = 155^\circ 27' 54''$, $c = 29^\circ 46' 8''$, $A = 90^\circ$.

6. $c = 73^\circ 41' 35''$, $B = 99^\circ 17' 33^\circ$, $A = 90^\circ$.

7. $B = 47^\circ 13' 42''$, $C = 126^\circ 40' 24''$, $A = 90^\circ$.

8. $a = 160^\circ$, $A = 150^\circ$, $C = 90^\circ$.

9. $b = 165^\circ$, $B = 112^\circ$, $C = 90^\circ$.

10. $a = 65^\circ 23' 12''$, $b = 65^\circ 23' 12''$, $C = 90^\circ$.

11. Solve $a = 90^\circ$, $B = 75^\circ 42'$, $c = 18^\circ 37'$. This is a quadrantal triangle, one side is a quadrant. The polar triangle is a right spherical triangle. Solve the right triangle and pass back to the original triangle.

12. Solve $a = 90^\circ$, $C = 42^\circ 10'$, $A = 115^\circ 20'$.

Art. XXIV. Solution of Oblique Spherical Triangles.

An oblique spherical triangle is determined when any three parts are given.

As in oblique plane triangles, provision must be made for three cases — when two of the three given parts are opposite each other, when no two of the three given parts are adjacent, and when the three given parts are adjacent.

64. Formulas when Two Parts Are Opposite. — In this case the sine formula is used,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

65. Formulas when no Parts Are Adjacent. — In this case the three sides or the three angles are given.

The formula

$$(a)\qquad \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

is not adapted to computation by logarithms. To adapt this formula to computation by logarithms subtract both sides of (a) from unity and simplify. There results

$$1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c} = \frac{\cos (b - c) - \cos a}{\sin b \sin c}$$

$$= \frac{2 \sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c}.$$

Now placing $a + b + c = 2s$, whence $a + b - c = 2s - 2c$, $a - b + c = 2s - 2b$, and remembering that

$$1 - \cos A = 2 \sin^2 (\tfrac{1}{2}A), \quad \sin^2 (\tfrac{1}{2}A) = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}.$$

Extracting the square root and interchanging letters in the result, three formulas adapted to computation by logarithms are obtained,

$$\mathbf{\sin \tfrac{1}{2}A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}}$$

$$(27)\qquad \mathbf{\sin \tfrac{1}{2}B = \sqrt{\frac{\sin (s - a) \sin (s - c)}{\sin a \sin c}}}$$

$$\mathbf{\sin \tfrac{1}{2}C = \sqrt{\frac{\sin (s - a) \sin (s - b)}{\sin a \sin b}}}.$$

Adding both sides of (a) to unity and proceeding as above there results

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}$$

(28) $$\cos \tfrac{1}{2}B = \sqrt{\frac{\sin s \sin (s-b)}{\sin a \sin c}}$$

$$\cos \tfrac{1}{2}C = \sqrt{\frac{\sin s \sin (s-c)}{\sin b \sin a}}.$$

Dividing (27) by (28)

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}$$

(29) $$\tan \tfrac{1}{2}B = \sqrt{\frac{\sin (s-a) \sin (s-c)}{\sin s \sin (s-b)}}$$

$$\tan \tfrac{1}{2}C = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)}}.$$

Formulas (29) may be written

$$\tan \tfrac{1}{2}\mathbf{A} = \frac{\mathbf{k}}{\sin (\mathbf{s}-\mathbf{a})}, \quad \tan \tfrac{1}{2}\mathbf{B} = \frac{\mathbf{k}}{\sin (\mathbf{s}-\mathbf{b})},$$

(30) $$\tan \tfrac{1}{2}\mathbf{C} = \frac{\mathbf{k}}{\sin (\mathbf{s}-\mathbf{c})},$$

where

$$\mathbf{k} = \sqrt{\frac{\sin (\mathbf{s}-\mathbf{a}) \sin (\mathbf{s}-\mathbf{b}) \sin (\mathbf{s}-\mathbf{c})}{\sin \mathbf{s}}}.$$

Applying (27) to the polar triangle and placing

$$A + B + C = 2S$$

$$\cot \tfrac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{-\cos S\cos(S-A)}}$$

(31) $$\cot \tfrac{1}{2}b = \sqrt{\frac{\cos(S-A)\cos(S-C)}{-\cos S\cos(S-B)}}$$

$$\cot \tfrac{1}{2}c = \sqrt{\frac{\cos(S-A)\cos(S-B)}{-\cos S\cos(S-C)}}.$$

These formulas may be written

$$\cot \tfrac{1}{2}\mathbf{a} = \frac{\mathbf{K}}{\cos(\mathbf{S}-\mathbf{A})},\quad \cot \tfrac{1}{2}\mathbf{b} = \frac{\mathbf{K}}{\cos(\mathbf{S}-\mathbf{B})},$$

(32) $$\cot \tfrac{1}{2}\mathbf{c} = \frac{\mathbf{K}}{\cos(\mathbf{S}-\mathbf{C})},$$

where

$$\mathbf{K} = \sqrt{\frac{\cos(\mathbf{S}-\mathbf{A})\cos(\mathbf{S}-\mathbf{B})\cos(\mathbf{S}-\mathbf{C})}{-\cos \mathbf{S}}}.$$

66. Formulas When Three Parts Are Adjacent. — Substituting from (27) and (28) in the formula

$$\cos\tfrac{1}{2}(A+B) = \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B - \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B$$

and simplifying

$$\cos\tfrac{1}{2}(A+B) = \frac{\sin s - \sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}$$

$$= \frac{2\sin\tfrac{1}{2}c\cdot\cos\tfrac{1}{2}(a+b)}{2\sin\tfrac{1}{2}c\cdot\cos\tfrac{1}{2}c}\sin\tfrac{1}{2}C = \frac{\cos\tfrac{1}{2}(a+b)}{\cos\tfrac{1}{2}c}\sin\tfrac{1}{2}C.$$

In like manner

$$\cos\tfrac{1}{2}(A-B) = \frac{\sin\tfrac{1}{2}(a+b)}{\sin\tfrac{1}{2}c}\sin\tfrac{1}{2}C,$$

$$\sin\tfrac{1}{2}(A+B) = \frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}c}\cos\tfrac{1}{2}C,$$

$$\sin\tfrac{1}{2}(A-B) = \frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}c}\cos\tfrac{1}{2}C.$$

These four equations are known as Gauss' equations.

67. From Gauss' equations are obtained the following four formulas, known as Napier's analogies,

$$
(33)\qquad
\begin{aligned}
\tan\tfrac{1}{2}(\mathbf{A}+\mathbf{B}) &= \frac{\cos\frac{1}{2}(\mathbf{a}-\mathbf{b})}{\cos\frac{1}{2}(\mathbf{a}+\mathbf{b})}\cot\tfrac{1}{2}\mathbf{C},\\
\tan\tfrac{1}{2}(\mathbf{A}-\mathbf{B}) &= \frac{\sin\frac{1}{2}(\mathbf{a}-\mathbf{b})}{\sin\frac{1}{2}(\mathbf{a}+\mathbf{b})}\cot\tfrac{1}{2}\mathbf{C},\\
\tan\tfrac{1}{2}(\mathbf{a}+\mathbf{b}) &= \frac{\cos\frac{1}{2}(\mathbf{A}-\mathbf{B})}{\cos\frac{1}{2}(\mathbf{A}+\mathbf{B})}\tan\tfrac{1}{2}\mathbf{c},\\
\tan\tfrac{1}{2}(\mathbf{a}-\mathbf{b}) &= \frac{\sin\frac{1}{2}(\mathbf{A}-\mathbf{B})}{\sin\frac{1}{2}(\mathbf{A}+\mathbf{B})}\tan\tfrac{1}{2}\mathbf{c}.
\end{aligned}
$$

The first of Napier's analogies may be stated as follows: In any spherical triangle the tangent of half the sum of any two angles equals the ratio of the cosine of half the difference of the sides opposite these angles to the cosine of half the sum of these sides multiplied by the cotangent of half the third angle.

State the remaining three analogies in the same way.

The formulas (30), (32), (33), together with the sine formula are sufficient to solve all oblique spherical triangles.

68. Example I. — Given

$$a = 103^\circ\,10', \quad b = 120^\circ\,12', \quad B = 131^\circ\,49',$$

find A, C and c.

The formula $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b}$ determines A; c and C are then found by the analogies,

$$\tan\tfrac{1}{2}c = \frac{\cos\frac{1}{2}(B+A)}{\cos\frac{1}{2}(B-A)}\tan\tfrac{1}{2}(b+a),$$

$$\cot\tfrac{1}{2}C = \frac{\cos\frac{1}{2}(b+a)}{\cos\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(B+A).$$

Arrange the work as follows :

$$\begin{array}{rr} \log\sin a = & 9.98843 \\ +\log\sin B = & \underline{9.87334} \\ & 19.86177 \\ -\log\sin b = & \underline{9.93665} \\ \log\sin A = & 9.92512 \end{array}$$

$$A = 57^\circ\,18'\,45'' \text{ or } 122^\circ\,41'\,15''.$$

Neither value of A when substituted in the analogies leads to a contradiction of signs, hence both values of A can be used and the triangle has two solutions.

When $A = 57^\circ\,18'\,45''$

$$\begin{array}{rr} \log\cos\frac{1}{2}(B+A) = & 8.89363 \\ +\log\tan\frac{1}{2}(b+a) = & \underline{.40054} \\ & 9.29417 \\ -\log\cos\frac{1}{2}(B-A) = & \underline{9.90133} \\ \log\tan\frac{1}{2}c = & 9.39284 \end{array}$$

$$\therefore c = 27^\circ\,45'\,26''$$

$$\begin{array}{rr} \log\cos\frac{1}{2}(b+a) = & 9.56759 \\ +\log\tan\frac{1}{2}(B+A) = & \underline{1.10504} \\ & 10.67263 \\ -\log\cos\frac{1}{2}(b-a) = & \underline{9.90518} \\ \log\cot\frac{1}{2}C = & 0.76745 \end{array}$$

$$\therefore C = 23^\circ\,44'\,14''$$

When $A = 122^\circ\,41'\,15''$

$$\begin{array}{rr} \log\cos\frac{1}{2}(B+A) = & 9.78123 \\ +\log\tan\frac{1}{2}(b+a) = & \underline{.40054} \\ & 10.18177 \\ -\log\cos\frac{1}{2}(B-A) = & \underline{9.99867} \\ \log\tan\frac{1}{2}c = & .18310 \end{array}$$

$$\therefore c = 113^\circ\,28'\,14''$$

$$
\begin{aligned}
\log\cos\tfrac{1}{2}(b+a) &= 9.56750 \\
+\log\tan\tfrac{1}{2}(B+A) &= \underline{.12010} \\
&\ 9.68769 \\
-\log\cos\tfrac{1}{2}(b-a) &= \underline{9.99518} \\
\log\cot\tfrac{1}{2}C &= 9.60251 \\
\therefore C &= 127^\circ\,32'\,56''
\end{aligned}
$$

Verify the results by applying the formula

$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}.$$

For $A = 57^\circ\,18'\,45''$

$$
\begin{aligned}
\log\sin A &= 9.92512 & \log\sin a &= 9.98843 \\
+\log\sin c &= \underline{9.66813} & +\log\sin C &= \underline{9.60481} \\
&\ 19.59325 & &\ 19.59324
\end{aligned}
$$

For $A = 122^\circ\,41'\,15''$

$$
\begin{aligned}
\log\sin A &= 9.62512 & \log\sin a &= 9.98843 \\
+\log\sin c &= \underline{9.96250} & +\log\sin C &= \underline{9.89918} \\
&\ 19.88762 & &\ 19.88761
\end{aligned}
$$

69. Example II.—Given $A = 135^\circ\,5'\,28''$, $C = 50^\circ\,30'\,8''$, $b = 69^\circ\,34'\,56''$, find a, c and B.

Compute a and c by using the analogies

$$\tan\tfrac{1}{2}(a+c) = \frac{\cos\frac{1}{2}(A-C)}{\cos\frac{1}{2}(A+C)}\tan\tfrac{1}{2}b,$$

$$\tan\tfrac{1}{2}(a-c) = \frac{\sin\frac{1}{2}(A-C)}{\sin\frac{1}{2}(A+C)}\tan\tfrac{1}{2}b,$$

and B by using

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

Since $\tan\frac{1}{2}(a+c)$ is negative, $\frac{1}{2}(a+c)$ is greater than 90°.

Arrange the work as follows :

$$\begin{aligned}
\log\cos\tfrac{1}{2}(A-C) &= 9.86905\\
+\log\tan\tfrac{1}{2}b &= \underline{9.84185}\\
&\ 9.71090\\
-\log\cos\tfrac{1}{2}(A+C) &= \underline{8.68837}\\
\log\tan\tfrac{1}{2}(a+c) &= 1.02253\\
\therefore\ \tfrac{1}{2}(a+c) &= 180^\circ - 84^\circ\,34'\,34''\\
&= \phantom{180^\circ - {}} 95^\circ\,25'\,25''
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(A-C) &= 9.82798\\
+\log\tan\tfrac{1}{2}b &= \underline{9.84185}\\
&\ 9.66983\\
-\log\sin\tfrac{1}{2}(A+C) &= \underline{9.99948}\\
\log\sin\tfrac{1}{2}(a-c) &= 9.67035\\
\therefore\ \tfrac{1}{2}(a-c) &= 25^\circ\,5'\,5''
\end{aligned}$$

Whence

$$a = \tfrac{1}{2}(a+c) + \tfrac{1}{2}(a-c) = 120^\circ\,30'\,30''$$

and

$$c = \tfrac{1}{2}(a+c) - \tfrac{1}{2}(a-c) = 70^\circ\,20'\,20''.$$

$$\begin{aligned}
\log\sin b &= 9.97182\\
+\log\sin C &= \underline{9.88742}\\
&\ 9.85924\\
-\log\sin c &= \underline{9.97391}\\
\log\sin B &= 9.88533\\
\therefore\ B &= 50^\circ\,10'\,10'' \quad\text{or}\quad 129^\circ\,49'\,50''
\end{aligned}$$

Since the greater angle must lie opposite the greater side, the angle $B = 50^\circ\,10'\,10''$.

Check the results by the analogy

$$\tan\tfrac{1}{2}(A+B) = \frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C$$

$$\begin{array}{rr}
\log\tan\frac{1}{2}(A+B) = & 1.33773 \\
+\log\cos\frac{1}{2}(a+b) = & \underline{8.94420} \\
 & 10.28193
\end{array}$$

$$\begin{array}{rr}
\log\cos\frac{1}{2}(a-b) = & 9.05562 \\
+\log\cot\frac{1}{2}C = & \underline{.32638} \\
 & 10.28200
\end{array}$$

The algebraic signs in the check formula agree.

70. Example III.—Given $A = 109°\,55'\,42''$, $B = 116°\,38'\,33''$, $C = 120°\,43'\,37''$, find a, b, c.

Use the formulas

$$\cot\frac{1}{2}a = \frac{K}{\cos(S-A)}, \quad \cot\frac{1}{2}b = \frac{K}{\cos(S-B)},$$

$$\cot\frac{1}{2}c = \frac{K}{\cos(S-C)},$$

$$K = \sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}}.$$

Arrange the work as follows :

$S = \frac{1}{2}(A+B+C) = 173°\,38'\,56''$, $S - A = 63°\,43'\,14''$, $S - B = 57°\,0'\,23''$, $S - C = 52°\,55'\,19''$.

Notice that $\cos S$ is negative and therefore K is real

$$\begin{array}{rr}
\log\cos(S-A) = & 9.64615 \\
+\log\cos(S-B) = & 9.73603 \\
+\log\cos(S-C) = & \underline{9.78025} \\
 & 29.16244 \\
\log\cos S = & \underline{9.99732} \\
2\log K = & 19.16512 \\
\log K = & 9.58256
\end{array}$$

Whence

$$\begin{aligned}\log\cot\tfrac{1}{2}a &= \log K - \log\cos(S-A) = 9.93640\\ \log\cot\tfrac{1}{2}b &= \log K - \log\cos(S-B) = 9.84653\\ \log\cot\tfrac{1}{2}c &= \log K - \log\cos(S-C) = 9.80231\end{aligned}$$

$$\begin{aligned}a &= 98^\circ\, 21'\, 38''\\ b &= 109^\circ\, 50'\, 20''\\ c &= 113^\circ\, 13'\, 28''\end{aligned}$$

Check the results by the formula $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b}$.

$$\begin{array}{rr@{\qquad}rr}\log\sin A = & 9.97318 & \log\sin B = & 9.95126\\ +\log\sin b = & \underline{9.97343} & +\log\sin c = & \underline{9.99536}\\ & 19.94661 & & 19.94662\end{array}$$

Problems. —

1. Given $a = 121^\circ\, 17'\, 18''$, $b = 76^\circ\, 31'\, 18''$, $C = 50^\circ\, 12'\, 12''$, show that $A = 133^\circ\, 55'\, 54''$, $B = 55^\circ\, 2'\, 12''$, $C = 50^\circ\, 12'\, 12''$.

2. Given $A = 78^\circ\, 19'$, $B = 36^\circ\, 15'$, $c = 112^\circ\, 38'$, show that $a = 101^\circ\, 30'$ $b = 36^\circ\, 16'\, 30''$, $C = 112^\circ\, 42'\, 54''$.

3. Given $a = 67^\circ\, 35'\, 12''$, $b = 58^\circ\, 36'\, 6''$, $A = 101^\circ\, 17'\, 48''$, show that $B = 64^\circ\, 52'\, 50''$, $c = 27^\circ\, 1'\, 58''$, $C = 29^\circ\, 54'\, 10''$.

4. Given $A = 39^\circ\, 45'$, $B = 26^\circ\, 59'$, $c = 154^\circ\, 37'$, show that $a = 121^\circ\, 27'\, 30''$, $b = 370^\circ\, 15'\, 30''$, $C = 161^\circ\, 22'$.

5. Given $A = 66^\circ\, 30'$, $B = 75^\circ\, 20'$, $C = 110^\circ\, 10'$, show that $a = 69^\circ\, 56'\, 30''$, $b = 82^\circ\, 16'\, 28''$, $c = 105^\circ\, 57'$.

6. Given $a = 112^\circ\, 6'\, 42''$, $b = 127^\circ\, 39'\, 12''$, $C = 71^\circ\, 12'\, 42''$, show that $A = 103^\circ\, 52'$, $B = 123^\circ\, 56'$, $C = 82^\circ\, 47'$.

7. Given $a = 42^\circ\, 45'$, $b = 47^\circ\, 15'$, $A = 56^\circ\, 30'$, show that $B = 64^\circ\, 26'\, 4''$, $C = 77^\circ\, 21'\, 36''$, $c = 52^\circ\, 35'\, 18''$, or $B = 115^\circ\, 33'\, 56''$, $C = 11^\circ\, 11'\, 42''$, $c = 9^\circ\, 5'\, 36''$.

Solve the following triangles :

8. $b = 137°$, $c = 116°$, $A = 68° 33' 45''$.

9. $A = 101° 17' 48''$, $B = 64° 52' 50''$, $c = 28° 1' 58''$.

10. $a = 99° 40' 48''$, $b = 64° 23' 15''$, $A = 95° 38' 4''$.

11. $A = 129° 5' 28''$, $B = 142° 12' 42''$, $C = 105° 8' 10''$.

12. $A = 128° 45'$, $C = 30° 35'$, $a = 68° 50'$.

13. $a = 50° 45' 20''$, $b = 69° 12' 40''$, $A = 44° 22' 10''$.

14. $b = 99° 40' 48''$, $c = 100° 49' 30''$, $A = 65° 33' 10''$.

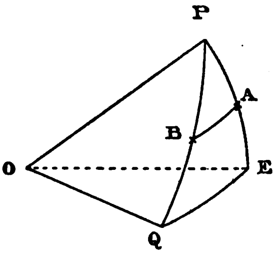
15. $a = 56° 40'$, $b = 83° 13'$, $c = 114° 30'$.

16. $A = 56° 12'$, $B = 58° 8'$, $a = 62° 42'$.

Art. XXV. Applications.

I. To find *the distance and bearing of two places A* and *B* on the surface of the earth, when the latitude and longitude of the places are given.

The earth is to be considered a sphere whose radius is 3960



miles, and the distance between the places is to be measured along a great circle arc connecting the two places.

Let O represent the center of the earth, P the north pole, EQ the equator, PE and PQ the meridians of the two places.

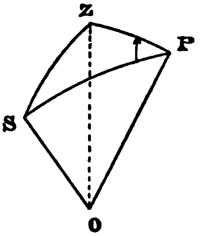
In the triangle PAB the angle P is the difference of the longitudes of A and B, the sides PA and PB are the complements of the latitudes of A and B respectively. The triangle PAB can therefore be solved. The arc AB becomes known in degrees and its length in miles can be computed. The angle PBA is the bearing east of north of A and B.

Problem. — Find the distance and bearing of San Francisco, latitude 37° 48′ N., longitude 122° 28′ W., from New York, latitude, 40° 43′ N., longitude 74° W.

II. To find *the time of day* at any place whose latitude is known, by observation of the sun.

The declination of the sun, that is the angular distance of the sun from the equator, is taken from the table in the nautical almanac.

The altitude of the sun, that is the angular distance of the sun from the horizon must be measured.



Let O represent the observer's position, the center of the celestial sphere; P the north pole of the heavens; Z the observer's zenith, that is the point vertically overhead; S the position of the sun at the instant the time is to be determined. The sun, and consequently the arc PS, makes one complete revolution about the pole P in twenty-four hours. That is a revolution of 360° takes twenty-four hours time. The great circle arc PZ is the observer's meridian, that is when the sun is on the great circle *PZ* and above the horizon it is 12 o'clock noon. If then the angle *ZPS* can be computed the time of day, before or after noon, becomes known.

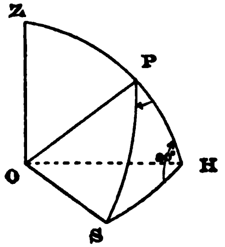
Since the line *OZ* is the direction of the observer's plumb line, and the observer's latitude is the angle which his plumb line makes with the equator, the angle *ZOP* is the complement of the latitude.

Now in the triangle *ZPS*, *PS* is the complement of the sun's declination ; *ZS* is the complement of the sun's altitude, since the zenith is at a quadrant's distance from the horizon ; and *PZ* is the complement of the observer's latitude. Hence the three sides of the triangle *ZPS* are known, and the angle *P* can be computed.

Problem. — In New York, latitude 40° 43′ N., in the forenoon of a day when the sun's declination is 10° N., what is the time of day ?

III. To find *the time of sunrise* at any place whose latitude is known on any day of the year.

The sun's declination is taken from the nautical almanac.



Let *O* represent the observer's position, *Z* his zenith, *HS* his horizon, *ZPH* his meridian. When the sun is on the meridian below the horizon it is midnight. The angle *SPH* reduced to time is the time after midnight when the sun reaches the horizon, that is, the time of sunrise. In the triangle *PHS* the angle

H is a right angle; *PH* is the observer's latitude; *PS* is the complement of the sun's declination. Hence the angle *P* can be computed and the time of sunrise becomes known.

Problem. — Find the time of sunrise in New York, latitude 40° 43′ N., when the sun's declination is 4° 30′ N.

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